Differential Equations

Linear, Nonlinear, Ordinary, Partial

When mathematical modelling is used to describe physical, biological or chemical phenomena, one of the most common results of the modelling process is a system of ordinary or partial differential equations. Finding and interpreting the solutions of these differential equations is therefore a central part of applied mathematics, and a thorough understanding of differential equations is essential for any applied mathematician. The aim of this book is to develop the required skills on the part of the reader.

The authors focus on the business of constructing solutions analytically and interpreting their meaning, although they do use rigorous analysis where needed. The reader is assumed to have some basic knowledge of linear, constant coefficient ordinary differential equations, real analysis and linear algebra. The book will thus appeal to undergraduates in mathematics, but would also be of use to physicists and engineers. MATLAB is used extensively to illustrate the material. There are many worked examples based on interesting real-world problems. A large selection of exercises is provided, including several lengthier projects, some of which involve the use of MATLAB. The coverage is broad, ranging from basic second-order ODEs including the method of Frobenius, Sturm-Liouville theory, Fourier and Laplace transforms, and existence and uniqueness, through to techniques for nonlinear differential equations including phase plane methods, bifurcation theory and chaos, asymptotic methods, and control theory. This broad coverage, the authors' clear presentation and the fact that the book has been thoroughly class-tested will increase its appeal to undergraduates at each stage of their studies.

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Preface

When mathematical modelling is used to describe physical, biological or chemical phenomena, one of the most common results is either a differential equation or a system of differential equations, together with appropriate boundary and initial conditions. These differential equations may be ordinary or partial, and finding and interpreting their solution is at the heart of applied mathematics. A thorough introduction to differential equations is therefore a necessary part of the education of any applied mathematician, and this book is aimed at building up skills in this area. For similar reasons, the book should also be of use to mathematically-inclined physicists and engineers.

Although the importance of studying differential equations is not generally in question, exactly how the theory of differential equations should be taught, and what aspects should be emphasized, is more controversial. In our experience, textbooks on differential equations usually fall into one of two categories. Firstly, there is the type of textbook that emphasizes the importance of abstract mathematical results, proving each of its theorems with full mathematical rigour. Such textbooks are usually aimed at graduate students, and are inappropriate for the average undergraduate. Secondly, there is the type of textbook that shows the student how to construct solutions of differential equations, with particular emphasis on algorithmic methods. These textbooks often tackle only linear equations, and have no pretension to mathematical rigour. However, they are usually well-stocked with interesting examples, and often include sections on numerical solution methods.

In this textbook, we steer a course between these two extremes, starting at the level of preparedness of a typical, but well-motivated, second year undergraduate at a British university. As such, the book begins in an unsophisticated style with the clear objective of obtaining quantitative results for a particular linear ordinary differential equation. The text is, however, written in a progressive manner, with the aim of developing a deeper understanding of ordinary and partial differential equations, including conditions for the existence and uniqueness of solutions, solutions by group theoretical and asymptotic methods, the basic ideas of control theory, and nonlinear systems, including bifurcation theory and chaos. The emphasis of the book is on analytical and asymptotic solution methods. However, where appropriate, we have supplemented the text by including numerical solutions and graphs produced using MATLAB†, version 6. We assume some knowledge of

† MATLAB is a registered trademark of The MathWorks, Inc.

MATLAB (summarized in Appendix 7), but explain any nontrivial aspects as they arise. Where mathematical rigour is required, we have presented the appropriate analysis, on the basis that the student has taken first courses in analysis and linear algebra. We have, however, avoided any functional analysis. Most of the material in the book has been taught by us in courses for undergraduates at the University of Birmingham. This has given us some insight into what students find difficult, and, as a consequence, what needs to be emphasized and re-iterated.

The book is divided into two parts. In the first of these, we tackle linear differential equations. The first three chapters are concerned with variable coefficient, linear, second order ordinary differential equations, emphasizing the methods of reduction of order and variation of parameters, and series solution by the method of Frobenius. In particular, we discuss Legendre functions (Chapter 2) and Bessel functions (Chapter 3) in detail, and motivate this by giving examples of how they arise in real modelling problems. These examples lead to partial differential equations, and we use separation of variables to obtain Legendre's and Bessel's equations. In Chapter 4, the emphasis is on boundary value problems, and we show how these differ from initial value problems. We introduce Sturm–Liouville theory in this chapter, and prove various results on eigenvalue problems. The next two chapters of the first part of the book are concerned with Fourier series, and Fourier and Laplace transforms. We discuss in detail the convergence of Fourier series, since the analysis involved is far more straightforward than that associated with other basis functions. Our approach to Fourier transforms involves a short introduction to the theory of generalized functions. The advantage of this approach is that a discussion of what types of function possess a Fourier transform is straightforward, since *all* generalized functions possess a Fourier transform. We show how Fourier transforms can be used to construct the free space Green's function for both ordinary and partial differential equations. We also use Fourier transforms to derive the solutions of the Dirichlet and Neumann problems for Laplace's equation. Our discussion of the Laplace transform includes an outline proof of the inversion theorem, and several examples of physical problems, for example involving diffusion, that can be solved by this method. In Chapter 7 we discuss the classification of linear, second order partial differential equations, emphasizing the reasons why the canonical examples of elliptic, parabolic and hyperbolic equations, namely Laplace's equation, the diffusion equation and the wave equation, have the properties that they do. We also consider complex variable methods for solving Laplace's equation, emphasizing their application to problems in fluid mechanics.

The second part of the book is concerned with nonlinear problems and more advanced techniques. Although we have used a lot of the material in Chapters 9 and 14 (phase plane techniques and control theory) in a course for second year undergraduates, the bulk of the material here is aimed at third year students. We begin in Chapter 8 with a brief introduction to the rigorous analysis of ordinary differential equations. Here the emphasis is on existence, uniqueness and comparison theorems. In Chapter 9 we introduce the phase plane and its associated techniques. This is the first of three chapters (the others being Chapters 13 and 15) that form an introduction to the theory of nonlinear ordinary differential equations, often known as dynamical systems. In Chapter 10, we show how the ideas of group theory can be used to find exact solutions of ordinary and partial differential equations. In Chapters 11 and 12 we discuss the theory and practice of asymptotic analysis. After discussing the basic ideas at the beginning of Chapter 11, we move on to study the three most important techniques for the asymptotic evaluation of integrals: Laplace's method, the method of stationary phase and the method of steepest descents. Chapter 12 is devoted to the asymptotic solution of differential equations, and we introduce the method of matched asymptotic expansions, and the associated idea of asymptotic matching, the method of multiple scales, including Kuzmak's method for analysing the slow damping of nonlinear oscillators, and the WKB expansion. We illustrate each of these methods with a wide variety of examples, for both nonlinear ordinary differential equations and partial differential equations. In Chapter 13 we cover the centre manifold theorem, Lyapunov functions and an introduction to bifurcation theory. Chapter 14 is about time-optimal control theory in the phase plane, and includes a discussion of the controllability matrix and the time-optimal maximum principle for second order linear systems of ordinary differential equations. Chapter 15 is on chaotic systems, and, after some illustrative examples, emphasizes the theory of homoclinic tangles and Mel'nikov theory.

There is a set of exercises at the end of each chapter. Harder exercises are marked with a star, and many chapters include a project, which is rather longer than the average exercise, and whose solution involves searches in the library or on the Internet, and deeper study. Bona fide teachers and instructors can obtain full worked solutions to many of the exercises by emailing solutions@cambridge.org.

In order to follow many of the ideas and calculations that we describe in this book, and to fully appreciate the more advanced material, the reader may need to acquire (or refresh) some basic skills. These are covered in the appendices, and fall into six basic areas: linear algebra, continuity and differentiability, power series, sequences and series of functions, ordinary differential equations and complex variables.

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