1
Introduction

1.1 Motivation

Interactions in many-body systems bring about collective phenomena such as superconductivity and magnetism. In many cases, simple mean-field theory provides a basic understanding of these phenomena. In fermion systems in one dimension, however, neither the mean-field theory nor perturbation theory works if it starts from the non-interacting fermions. This is because the interaction effects in one dimension are much stronger than those in higher dimensions. Intuitively speaking, two particles cannot avoid collision in a single-way track in contrast with two and three dimensions. Thus the interaction effects appear in a drastic way in one dimension.

Another aspect in one dimension, which overcompensates the difficulty of perturbation and mean-field theories, is that a complete account of interaction effects is possible under certain conditions. The class of systems satisfying such conditions is referred to as exactly solvable. Soon after the establishment of quantum mechanics, Bethe solved exactly the Heisenberg spin model in one dimension [28]. The basic idea of the solution is now called the Bethe ansatz. Since then, theoretical physics in one dimension has developed into a magnificent edifice, including sophisticated mathematical techniques. In many cases, the eigenfunctions derived by the Bethe ansatz consist of plane waves that are defined stepwise for each spatial configuration of particles. Since the coefficients of plane waves depend on the configuration, the property of the wave function cannot be made explicit without detailed knowledge of these coefficients. We mention some of the recent monographs on the Bethe ansatz and its extensions [54, 118, 179]. A comprehensive account on exactly solvable models has recently been given by Sutherland [178].

The models solved by the Bethe ansatz are characterized by short-range interactions such as on-site repulsion or the next-nearest-neighbor exchange interaction. On the other hand, it was found by Calogero that another class of models also permits exact solution [34, 35]. The models have repulsive interactions decaying as the inverse square of the interparticle distance r. In order to prevent the blow-up of particles toward infinite distance, an attractive harmonic potential can be added to the system. Alternatively, one takes the periodic boundary condition with the system length L, and employs superposition of the $1/r^2$ potential as

$$\sum_{n=-\infty}^{\infty} \frac{1}{(r+nL)^2} = \left(\frac{\pi/L}{\sin \pi r/L}\right)^2. \tag{1.1}$$

Then by construction the system does not blow up, while keeping the translational invariance. This model was proposed by Sutherland [172, 174], and hence is called the Sutherland model. If one refers to both models simultaneously, it seems appropriate to call them the Calogero–Sutherland models. Some years later, Moser analyzed the classic mechanical version of these models mathematically [135], and his name is sometimes added in referring to the models.

The $1/r^2$ models have much simpler mathematical (algebraic) structure, compared to the conventional integrable models solved by the Bethe ansatz. This simplicity enables us to derive explicitly the exact expressions of dynamical correlation functions such as the Green function, the density–density correlation function, and the spin–spin correlation function. The resultant expressions are remarkably simple, but still keep nontrivial features inherent to interacting particle systems. Further, the mathematical tools used in the derivation are far from complicated. Thus, the $1/r^2$ models provide comprehensible examples for studying dynamics of interacting particles.

In contrast with the Fermi liquid in three dimensions, the one-dimensional fermions behave as the Tomonaga–Luttinger liquid in the limit of long time and long distance. Here the conformal field theory (CFT) describes nicely the asymptotics of correlation functions. According to the CFT, characterization of the interaction parameters can be done through analysis of the finite-size correction of the ground state energy. Since the $1/r^2$ models allow for calculation of the finite-size correction much more easily than the Bethe-solvable models, the $1/r^2$ models serve as an instructive example to visualize how the CFT works in the Tomonaga–Luttinger liquid. The importance of the $1/r^2$ models does not, however, lie only in the mathematical structure. Through the study of the $1/r^2$ models, one can also learn about the dynamics of the correlated electrons in real systems. For example, the neutron scattering intensity of $S = 1/2$ antiferromagnetic spin chain reveals a similarity to the spectral function of the spin correlation function of the $1/r^2$ exchange interaction model, which is called the Haldane–Shastry model [77, 161]. A related model with charge degrees of freedom is still exactly solvable provided a supersymmetry is imposed [119]. The spin–charge separation of one-dimensional electrons can then be explicitly seen in the spectral weight of the Green function of the supersymmetric t–J model.

1.2 One-dimensional interaction as a disguise

As we shall explain in detail, the wave function in the ground state of the $1/r^2$ models can be derived explicitly as the product of two-body wave functions. This feature is quite in contrast with cases solved by the Bethe ansatz. The special feature of the $1/r^2$ interaction already appears in most elementary quantum mechanics. Let us consider a free particle with mass $m = 1/2$ in the three-dimensional space. The Hamiltonian is given by

$$H = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} = -\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{\boldsymbol{l}^2}{r^2}, \tag{1.2}$$

where $r^2 = x^2 + y^2 + z^2$, and $\boldsymbol{l}$ is the angular momentum operator. We take the units $\hbar = 1$ throughout the book. In the polar coordinates, there appears a fictitious potential leading to the centrifugal force. Namely, the free motion in higher dimensions generates a fictitious potential if the radial motion alone is extracted [146]. Conversely, the potential $l(l+1)/r^2$ in the radial coordinate is a disguise of free motion in higher dimensions. The form of the radial kinetic energy in (1.2) is interpreted as coming from the metric of the one-dimensional space. Pursuing this idea in many-body systems, one gains a perspective that interactions in exactly solvable models are a disguise of some kind of free motion in another space [146]. Alternatively, a matrix model has been constructed where the coordinates of N particles are regarded as eigenvalues of an $N \times N$ matrix. The transformation matrix for diagonalization appears as the $1/r^2$ potential [151].

In the early stage of the Tomonaga–Luttinger theory, all low-energy excitations are regarded as bosons. Actually, the statistics of excitations need not be restricted to bosons. In some cases, the interaction among bosons is absorbed into a new statistics describing exclusion of available one-body states. This idea applies to many interacting systems approximately, and to the $1/r^2$ models exactly. The exclusion includes fermions and bosons as special cases. Generally, however, the statistics is fractional. In order to account for the resultant quasi-particles obeying fractional exclusion statistics, concepts such as the Yangian symmetry and the supersymmetry turn out to be useful. These new concepts make it much easier to understand exact dynamics (and also thermodynamics) intuitively. Our key strategy in this respect is to rely on the picture of quasi-particles obeying fractional exclusion statistics. In terms of these exotic quasi-particles, the dynamics of one-dimensional systems can be understood intuitively.

In the last decades, intensive study of the $1/r^2$ models has brought about deep intuition into the structure of the excitation spectrum in one-dimensional systems in general. The most remarkable observation is that elementary excitations behave as free particles subject to certain statistical constraints. As a result, these particles obey the statistics of neither fermions nor bosons. In other words, the exchange of two excitations leads to a scattering phase shift which is independent of their momenta, but which is neither π (antisymmetric) nor 0 (symmetric).

The situation may become clearer if we make an analogy to the Fermi liquid theory. The excitations in the Bethe-soluble models have a phase shift that does depend on their momenta. Therefore, certain parameters are necessary to characterize the momentum dependence. These parameters are analogous to Landau parameters that describe interactions between the quasi-particles in the Fermi liquid. In this analogy, the excitations in the $1/r^2$ models do not need the analogue of the Landau parameters, and are comparable to free fermions except for the statistics. Just as the understanding of metals in general has been much facilitated by the free-electron model, the dynamics in one dimension should be much better understood by reference to "free" models, i.e., the $1/r^2$ models.

1.3 Two-body problem with $1/r^2$ interaction

We demonstrate the peculiar features of the $1/r^2$ model by taking the simplest example. Let us consider the two-body problem with Hamiltonian

$$H_2 = -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} + g\left[\frac{\pi/L}{\sin\pi(x_1-x_2)/L}\right]^2. \tag{1.3}$$

For the moment we assume that the two particles are distinguishable, and do not care about the symmetry of the wave function. If the distance $|x_1 - x_2|$ is much smaller than L, the interaction reduces to $g/(x_1-x_2)^2$. The center of gravity $X = (x_1+x_2)/2$ has free motion with wave number Q. In terms of X and the relative coordinate $x = x_1 - x_2$, the wave function is factorized into the form $\psi_g(x_1, x_2) = \phi_g(x)\exp(\mathrm{i}QX)$, where $\phi_g(x)$ is an eigenfunction of a one-body Hamiltonian H_1 given by

$$H_1(x) = H_2 - \frac{1}{2}Q^2 = -2\frac{\partial^2}{\partial x^2} + g\left(\frac{\pi/L}{\sin \pi x/L}\right)^2. \tag{1.4}$$

Instead of solving (1.4) in the standard way, we discuss alternative ideas which are useful in generalizing to the many-body problem. Let us first examine the wave function $\phi_g(x)$ for $|x| \ll L$ where the potential in H_1 tends to g/x^2. Then $H_1(x)$ has the scaling property

$$H_1(ax) = a^{-2}H_1(x).$$

An eigenfunction should also have the scaling property for $x \sim 0$

$$\phi_g(ax) = a^{\lambda}\phi_g(x), \tag{1.5}$$

with certain number λ. The only solution with property (1.5) is the power-law function $\phi_g(x) = x^{\lambda}$. Upon differentiation twice, we obtain $\lambda(\lambda-1)\phi_g(x)/x^2$. By taking $\lambda(\lambda-1) = g/2$, the kinetic term cancels the potential term. Then $\phi_g(x)$ turns out to be the eigenfunction of H_1. Since we have $\lambda = (1 \pm \sqrt{1+2g})/2$, only the case of $g \geq -1/2$ is meaningful. Otherwise, the attractive potential causes the system to collapse as in the classical system, and the ground state cannot be defined. This situation has already been discussed by Landau and Lifshitz [122] and by Sutherland [172]. In the following we only consider the case $g > 0$, and take the positive λ as the relevant solution. We can extend the range of x so as to be consistent with the periodic boundary condition, simply by replacing x^{α} by $|\sin \pi x/L|^{\alpha}$.

It is possible to derive all the eigenvalues and eigenfunctions by using the factorization method [89], which has been refined under the name of "supersymmetric quantum mechanics" [192]. We introduce a variable $\eta \equiv \pi x/L$ and rewrite (1.4) as

$$H_1 = 2\left(\frac{L}{\pi}\right)^2\left[p_\eta^2 + W_\lambda(\eta)^2 + W'_\lambda(\eta) + \lambda^2\right] \equiv 2\left(\frac{L}{\pi}\right)^2\mathcal{H}_\lambda, \tag{1.6}$$

where $p_\eta = -\mathrm{i}\partial/\partial\eta$ and $W_\lambda(\eta) = \lambda\cot\eta$. Then $\mathcal{H}_\lambda$ takes a factorized form

$$\mathcal{H}_\lambda = (p_\eta - \mathrm{i}W_\lambda)(p_\eta + \mathrm{i}W_\lambda) + \lambda^2 \equiv A_\lambda^\dagger A_\lambda + \lambda^2. \tag{1.7}$$

An eigenfunction of $\mathcal{H}_\lambda$ is given by

$$\phi_\lambda(\eta) = \sin^\lambda\eta = \exp[U_\lambda(\eta)], \tag{1.8}$$

where we have introduced $U_\lambda(\eta) = \lambda\ln\sin\eta$. This gives $U'_\lambda(\eta) = W_\lambda(\eta)$, and it is evident that $A_\lambda\phi_\lambda(\eta) = 0$. Since $A_\lambda^\dagger A_\lambda$ is a non-negative operator, there

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are no states with lower energy. Hence, ϕ_λ gives the ground state of $\mathcal{H}_\lambda$ with energy λ^2.

We note the property

$$\begin{aligned} A_\lambda A_\lambda^\dagger &= p_\eta^2 + W_\lambda(\eta)^2 - W_\lambda'(\eta) = p_\eta^2 + \frac{\lambda(\lambda+1)}{\sin^2\eta} - \lambda^2 \\ &= A_{\lambda+1}^\dagger A_{\lambda+1} - \lambda^2, \end{aligned} \tag{1.9}$$

which corresponds to the shift $\lambda \to \lambda+1$ in $\mathcal{H}_\lambda$ together with subtracting the constant term λ^2. Combination of (1.7) and (1.9) makes it possible to derive all the excited states. Let us take the ground state $\phi_{\lambda+1}$ of $H_{\lambda+1}$ with the eigenvalue $(\lambda+1)^2$. Namely, we have

$$A_\lambda A_\lambda^\dagger \phi_{\lambda+1} = [(\lambda+1)^2 - \lambda^2]\phi_{\lambda+1}. \tag{1.10}$$

Applying $A_\lambda^\dagger$ from the left, we obtain

$$A_\lambda^\dagger A_\lambda A_\lambda^\dagger \phi_{\lambda+1} = [(\lambda+1)^2 - \lambda^2] A_\lambda^\dagger \phi_{\lambda+1}. \tag{1.11}$$

Thus the state $A_\lambda^\dagger \phi_{\lambda+1}$ proves to be an excited state of $A_\lambda^\dagger A_\lambda$.

We now explain briefly the idea of the supersymmetric quantum mechanics. We may treat the pair $A_\lambda A_\lambda^\dagger$ and $A_\lambda^\dagger A_\lambda$ as components of a 2×2 matrix:

$$\mathcal{H}_{\text{pair}} = \begin{pmatrix} A_\lambda^\dagger A_\lambda & 0 \\ 0 & A_\lambda A_\lambda^\dagger \end{pmatrix} = QQ^\dagger + Q^\dagger Q \equiv \{Q, Q^\dagger\}, \tag{1.12}$$

where

$$Q = \begin{pmatrix} 0 & 0 \\ A_\lambda & 0 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & A_\lambda^\dagger \\ 0 & 0 \end{pmatrix}. \tag{1.13}$$

The space of the 2×2 matrix can be regarded as a pseudo-spin spanned by the Pauli matrices. Here Q and $Q^\dagger$ have an analogy with spin-flips $s_\pm = s_x \pm \mathrm{i}s_y$. Alternatively, we may include the pseudo-fermion operators $f, f^\dagger$ by the identification

$$\frac{1}{2}(1-\sigma_z) = f^\dagger f. \tag{1.14}$$

Then the operators $Q, Q^\dagger$ in (1.12) are written as

$$Q = f^\dagger A_\lambda, \quad Q^\dagger = A_\lambda^\dagger f. \tag{1.15}$$

It is obvious that $Q^2 = (Q^\dagger)^2 = 0$ and

$$[\mathcal{H}_{\text{pair}}, Q] = [\mathcal{H}_{\text{pair}}, Q^\dagger] = 0. \tag{1.16}$$

The last equality means that $\mathcal{H}_{\rm pair}$ is invariant against the pseudo-spin rotation, and the conserved quantity Q is called the supercharge. In this framework, the degeneracy demonstrated by (1.11) is interpreted as a consequence of the supersymmetry. The use of $Q, Q^\dagger$ motivates us to refer to the factorization method as "supersymmetric quantum mechanics".

The operators A_λ and $A_\lambda^\dagger$ have the commutation rule

$$[A_\lambda, A_\lambda^\dagger] = -2W_\lambda. \tag{1.17}$$

In a special case of $W_\lambda = -x/2$, the commutation rule reduces to that of bosonic creation and annihilation operators. Hence, A_λ and $A_\lambda^\dagger$ can be regarded as a generalization of bosonic operators.

Now we iterate the procedure of increasing λ by unity to obtain all excited states. Let us use the fact $A_\lambda^\dagger A_\lambda = A_{\lambda-1}A_{\lambda-1}^\dagger + (\lambda-1)^2 - \lambda^2$ as derived from (1.9). After this substitution in (1.11), we multiply $A_{\lambda-1}^\dagger$ from the left to obtain

$$A_{\lambda-1}^\dagger A_{\lambda-1} A_{\lambda-1}^\dagger A_\lambda^\dagger \phi_{\lambda+1} = [(\lambda+1)^2 - (\lambda-1)^2] A_{\lambda-1}^\dagger A_\lambda^\dagger \phi_{\lambda+1}, \tag{1.18}$$

which shows that $A_{\lambda-1}^\dagger A_\lambda^\dagger \phi_{\lambda+1}$ is an excited state of $A_{\lambda-1}^\dagger A_{\lambda-1}$. This process can be iterated. The wave function $\phi_{n+1;\lambda+1} \equiv A_{\lambda-n}^\dagger \cdots A_{\lambda-1}^\dagger A_\lambda^\dagger \phi_{\lambda+1}$ with $n \geq 0$ satisfies the equation

$$A_{\lambda-n}^\dagger A_{\lambda-n} \phi_{n+1;\lambda+1} = [(\lambda+1)^2 - (\lambda-n)^2] \phi_{n+1;\lambda+1}. \tag{1.19}$$

Equivalently we obtain for $m \geq 1$

$$\mathcal{H}_\lambda \phi_{m;\lambda} = (\lambda+m)^2 \phi_{m;\lambda}. \tag{1.20}$$

We identify $\phi_{0;\lambda}$ as ϕ_λ to include the case of $m=0$ in the above. In this way we can derive all the excited states of $\mathcal{H}_\lambda$ starting from the ground state of $H_{\lambda'}$ with appropriate $\lambda' > \lambda$. Figure 1.1 shows the situation where the ordinate κ gives the energy as κ^2.

Conversely, starting from a free state $\phi_{m;0} = \sin m\eta$ at $\lambda = 0$, we can construct the eigenfunctions of $\mathcal{H}_\lambda$ as $\phi_{m;\lambda} = A_{\lambda-1} \cdots A_1 A_0 \phi_{m+\lambda;0}$. Figure 1.1 also shows this inverse direction of construction. Note that the spectrum of $\mathcal{H}_\lambda$ above the ground-state energy λ^2 is the same as that of the free system. To derive $\phi_{n;\lambda+n}$ explicitly, we use the relation $A_\lambda^\dagger = \exp(-U_\lambda) p_\eta \exp(U_\lambda)$ and obtain

$$\begin{aligned}
\phi_{n;\lambda+n} &= \exp(-U_{\lambda+1}) p_\eta \exp(U_{\lambda+1} - U_{\lambda+2}) \cdots p_\eta \exp(U_{\lambda+n}) \phi_{\lambda+n} \\
&= \exp(-U_\lambda) [\exp(-U_1) p_\eta]^n \exp(U_{\lambda+n}) \phi_{\lambda+n} \\
&= \phi_\lambda \times (1-y^2)^{-\lambda} \left(-\mathrm{i} \frac{\mathrm{d}}{\mathrm{d}y} \right)^n (1-y^2)^{\lambda+n},
\end{aligned} \tag{1.21}$$

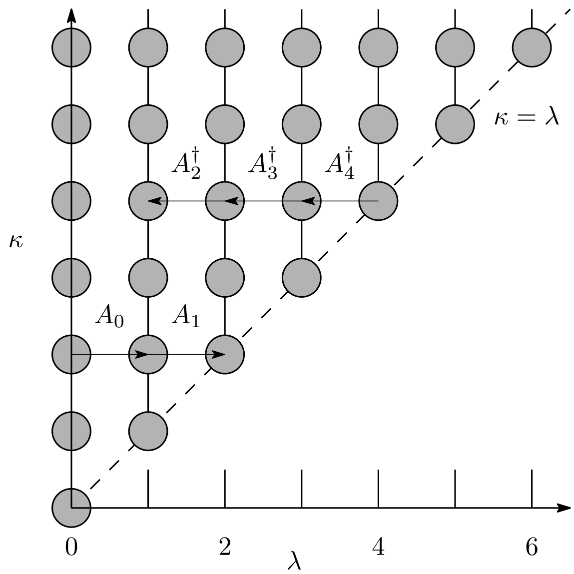

Fig. 1.1. The spectrum of the one-body Sutherland model $\mathcal{H}_\lambda$. The creation operator $A_\lambda^\dagger$ generates an excited state for $H_{\lambda-1}$ from an eigenstate of $\mathcal{H}_\lambda$. These two states are degenerate, and their energy is given by κ^2. The annihilation operator A_λ generates an eigenstate of $H_{\lambda+1}$ from that of $\mathcal{H}_\lambda$.

where $y = \cos\eta$. The last expression includes, apart from the normalization factor, the Rodrigues formula for the nth-order Gegenbauer polynomial $C_n^{\lambda+1/2}(y)$. Namely, we obtain

$$\phi_{n;\lambda+n}(\eta) = C_n^{\lambda+1/2}(\cos\eta)\phi_\lambda(\eta). \tag{1.22}$$

The generating function of Gegenbauer polynomials is given by

$$(1 - 2yt + t^2)^{-\lambda-1/2} = \sum_{n=0}^{\infty} C_n^{\lambda+1/2}(y)t^n. \tag{1.23}$$

In the special case $\lambda = 0$, it is reduced to the Legendre polynomial $C_n^{1/2}(y) = P_n(y)$. Because of the similarity to the Legendre polynomial, $C_n^{\lambda+1/2}(y)$ is also called the ultraspherical polynomial.

Let us come back to the two-body system. For each particle j, we introduce the complex coordinate $z_j = \exp(2\pi \mathrm{i} x_j/L)$ which specifies a point on the unit circle. The wave function of two particles is written as

$$\psi_{n;\lambda+n}(z_1, z_2) = \exp(\mathrm{i}QX)\phi_{n;\lambda+n}(\eta), \tag{1.24}$$

where $\eta = \pi(x_1 - x_2)/L$. We assert that $\psi_{n;\lambda+n}(z_1, z_2)$ is a homogeneous polynomial of z_1 and z_2 times an integer (or half-integer) power of z_1z_2.

The sum q of powers of z_1 and z_2 in $\psi_{n;\lambda+n}(z_1, z_2)$ is related to the total momentum Q as $q = LQ/(2\pi)$. To prove the assertion, we note the relations

$$2\cos\eta = (z_1/z_2)^{1/2} + (z_2/z_1)^{1/2} = (z_1 z_2)^{-1/2}(z_1 + z_2), \tag{1.25}$$

$$2\mathrm{i}\sin\eta = (z_1/z_2)^{1/2} - (z_2/z_1)^{1/2} = (z_1 z_2)^{-1/2}(z_1 - z_2), \tag{1.26}$$

$$\exp(\mathrm{i}QX) = (z_1 z_2)^{q/2}. \tag{1.27}$$

Since $\phi_{n;\lambda+n}(\eta)$ is a polynomial of $\cos\eta$ and $\sin\eta$, it is a polynomial of z_1 and z_2, times an integer or half-integer power of $z_1 z_2$. Thus the assertion is proved. The homogeneous polynomial of z_1 and z_2, which originates from a Gegenbauer polynomial of $\cos\eta$, corresponds to a special case of the Jack polynomial. The latter is defined for arbitrary number n of complex variables $z_1, z_2, \ldots, z_n$, as will be discussed in detail later.

We now proceed to the case of two identical (indistinguishable) particles. If the particles are bosons, we should take symmetric (even) wave functions. If the particles are spinless fermions, on the other hand, we take antisymmetric (odd) wave functions. For example,

$$\phi_\lambda(x) = |\sin\eta|^{\lambda-1}\sin\eta \tag{1.28}$$

describes the ground state of two fermions for H_2. If fermions have spin 1/2, the spatial part of the wave functions is either symmetric (spin singlet) or antisymmetric (spin triplet). In the case of bosons, the even–odd property becomes the opposite to that of fermions; exchange of spatial and spin coordinates at the same time gives the same wave function as that before the exchange. In order to discuss the case with internal degrees of freedom, we consider a generalized model as given by

$$H_{2;K} = -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} + \left(\frac{\pi}{L}\right)^2 \frac{2\lambda(\lambda - K_{12})}{\sin^2 \pi(x_1 - x_2)/L}, \tag{1.29}$$

where we have introduced the coordinate exchange operator K_{12}. As a complementary factor, we also introduce the spin permutation operator $P_{12} = 2\boldsymbol{S}_1 \cdot \boldsymbol{S}_2 + 1/2$. They act on a two-body wave function with spin coordinates σ_1, σ_2 as

$$K_{12}\psi(x_1, x_2; \sigma_1, \sigma_2) = \psi(x_2, x_1; \sigma_1, \sigma_2), \tag{1.30}$$

$$P_{12}\psi(x_1, x_2; \sigma_1, \sigma_2) = \psi(x_1, x_2; \sigma_2, \sigma_1), \tag{1.31}$$

$$P_{12}K_{12}\psi(x_1, x_2; \sigma_1, \sigma_2) = \psi(x_2, x_1; \sigma_2, \sigma_1) = \pm\psi(x_1, x_2; \sigma_1, \sigma_2). \tag{1.32}$$

Namely, we have $K_{12}P_{12} = \pm 1$ depending on whether the particles are bosons or fermions.

For symmetric spatial wave functions with $K_{12} = 1$, we know from (1.4) and (1.20) that the spectrum is given by

$$E_n = \frac{1}{2}Q^2 + 2\left(\frac{\pi}{L}\right)^2 (n+\lambda)^2 = k_+^2 + k_-^2, \tag{1.33}$$

where $k_\pm = [Q/2 \pm \pi(n+\lambda)/L]^2$ with $n \geq 0$. Note that E_n can be written as if it consists of the kinetic energy of free particles with momenta $k_\pm$. The interaction effect appears only in the restriction $k_+ - k_- \geq 2\pi\lambda/L$, which becomes the same as the Pauli exclusion principle in the case of $\lambda = 1$. For the antisymmetric (odd) wave function with $K_{12} = -1$, we have $\lambda(\lambda+1)$ in (1.29), and accordingly replace $\lambda \to \lambda + 1$ in (1.33). This is alternatively interpreted as taking the excitation level n one step higher. The odd-function ground state in particular is given by E_1, where $n = 1$ is the smallest degree of antisymmetric polynomials of z_1 and z_2 with $z_j = \exp(2\mathrm{i}\pi x_j/L)$. If there are spin degrees of freedom for a pair of fermions, the ground state energy becomes E_0 for the spin singlet, and E_1 for the triplet.

We have thus found that different symmetries of the wave functions appear only as a shift of energy levels. In particular, the difference between the singlet and triplet states in each ground state is given by

$$E_{\mathrm{g}}(S=1) - E_{\mathrm{g}}(S=0) = \pm\,(2\pi/L)^2\,(\lambda + 1/2), \tag{1.34}$$

where the plus sign is for fermions and the minus sign for bosons. The signs mean the antiferromagnetic interaction for fermions and the ferromagnetic interaction for bosons. With use of $K_{12}P_{12} = \pm 1$ for identical particles, we obtain the Hamiltonian equivalent to $H_{2;K}$ as

$$H_{2;P} = -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} + \left(\frac{L}{\pi}\right)^2 \frac{2\lambda(\lambda \pm P_{12})}{\sin^2 \pi(x_1 - x_2)/L}, \tag{1.35}$$

where the plus sign is for fermions and the minus sign for bosons. By construction, this model has both the charge and the spin degrees of freedom. One can extract only the spin degrees of freedom by taking a limiting procedure as explained next.

1.4 Freezing spatial motion

Let us consider the limiting case $\lambda \gg 1$ in $H_{2;K}$. Accordingly $\phi_\lambda(x)$ tends to a delta function peaked at $x = L/2$. This limit for a large number of particles is relevant to the spin chain and the supersymmetric (SUSY) t–J model since the particles should crystallize to avoid the repulsion, leaving