QUASI-FROBENIUS RINGS

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CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK
40 West 20th Street, New York, NY 10011-4211, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa

http://www.cambridge.org

First published 2003

Printed in the United Kingdom at the University Press, Cambridge

Typeface Times 10/13 pt. *System* LaTeX 2ε [TB]

A catalog record for this book is available from the British Library.

Library of Congress Cataloging in Publication Data
Nicholson, W. Keith.
Quasi-Frobenius rings / W. K. Nicholson, M. F. Yousif.
p. cm. – (Cambridge tracts in mathematics ; 158)
Includes bibliographical references and index.
ISBN 0-521-81593-2
1. Quasi-Frobenius rings. I. Yousif, M. F. (Mohamed F.) II. Title. III. Series.
QA251.5 .N53 2003
512′.4 – dc21 2002041003

ISBN 0 521 81593 2 hardback

Contents

List of Symbols

$\mathbb{N}$	Set of natural numbers
$\mathbb{Z}$	Ring of integers
$\mathbb{Q}$	Field of rational numbers
$\mathbb{R}$	Field of real numbers
$\mathbb{C}$	Field of complex numbers
$\mathbb{Z}_n$	Ring of integers modulo n
$\mathbb{Z}_{(p)}$	Integers localized at the prime p
$\mathbb{Z}_{p^\infty}$	Prüfer group at the prime p
δ_{ij}	Kronecker delta
$\lvert X\rvert$	Cardinality of a set X
$X \subset Y$	$X \subseteq Y$ and $X \neq Y$, for sets X and Y
$E(M)$	Injective hull of the module M
$soc(M)$	Socle of the module M
$rad(M)$	Radical of the module M
$dim(M)$	Uniform (Goldie) dimension of the module M
$length(M)$	Composition length of the module M
$Z(M)$	Singular submodule of the module M
$char(R)$	Characteristic of the ring R
$S_r,\ S_l$	$soc(R_R),\ soc({}_RR)$
$Z_r,\ Z_l$	$Z(R_R),\ Z({}_RR)$
$J,\ J(R)$	Jacobson radical of the ring R
$\mathtt{r}(X),\ \mathtt{l}(X)$	Left and right annihilators of the set X
$R[x]$	Polynomial ring over the ring R
$F(x)$	Ring of rational functions over the field F
$M_n(R)$	Ring of $n \times n$ matrices over the ring R
$R^n,\ R_n$	Row matrices, column matrices over the ring R
$end(M)$	Endomorphism ring of the module M
$K \subseteq^{ess} M$	K is an essential submodule of the module M

$K \subseteq^{max} M$	K is a maximal submodule of the module M
$K \subseteq^{sm} M$	K is a small submodule of the module M
$K \subseteq^{\oplus} M$	K is a direct summand of the module M
$c\cdot,\ \cdot c$	Left (right) multiplication map by the element c
$M^{(I)}$	The direct sum of $\lvert I\rvert$ copies of the module M
M^I	The direct product of $\lvert I\rvert$ copies of the module M
$lat(M)$	Lattice of submodules of the module M
M^*	Dual of the module M
$modR,\ Rmod$	Categories of right and left modules over the ring R
$V \otimes_R W,\ v \otimes w$	Tensor product of modules, elements

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Background

To make this monograph as self-contained as possible, this preliminary chapter contains basic characterizations of quasi-Frobenius and pseudo-Frobenius rings, together with the necessary background material. We assume familiarity with the basic facts of noncommutative ring theory, and we refer the reader to the texts by Anderson and Fuller [1] or Lam [131] for the relevant information. However, we make frequent use of facts about semiperfect, perfect, and semiregular rings and about Morita equivalence, often without comment. All these results are derived in the Appendices, again to make the book self-contained.

Throughout this book all rings considered are associative with unity and all R-modules are unital. We write $J = J(R)$ for the Jacobson radical of R and $M_n(R)$ for the ring of $n \times n$ matrices over R. Right and left modules are denoted M_R and ${}_RM$ respectively, and we write module homomorphisms opposite the scalars. If M is an R-module, we write $Z(M)$, $soc(M)$ and $M^* = hom_R(M, R)$ respectively, for the singular submodule, the socle, and the dual of M. The uniform (Goldie) dimension of a module M will be referred to simply as the dimension of M and will be denoted $dim(M)$. For a ring R, we write

$$soc(R_R) = S_r, \quad soc({}_RR) = S_l, \quad Z(R_R) = Z_r, \quad \text{and} \quad Z({}_RR) = Z_l.$$

The notations $N \subseteq^{max} M$, $N \subseteq^{ess} M$, and $N \subseteq^{sm} M$ mean that N is a maximal, (essential, and small) submodule of M, respectively, and we write $N \subseteq^{\oplus} M$ if N is a direct summand of M. Right annihilators will be denoted as

$$\mathtt{r}(Y) = \mathtt{r}_X(Y) = \{x \in X \mid yx = 0 \text{ for all } y \in Y\},$$

with a similar definition of left annihilators, $\mathtt{l}_X(Y) = \mathtt{l}(Y)$. Multiplication maps $x \mapsto ax$ and $x \mapsto xa$ will be denoted $a\cdot$ and $\cdot a$ respectively. If π is a property of modules, we say that M is a π *module* if it has the property π and that the ring R is a right π *ring* if R_R is a π module (with a similar convention on the left).

1.1. Injective Modules

Injective modules are closely related to essential extensions. If $K \subseteq M$ are modules, recall that K is called an *essential submodule* of M (and $K \subseteq M$ is called an *essential extension*) if $K \cap X \neq 0$ for every submodule $X \neq 0$ of M. This state of affairs is denoted $K \subseteq^{ess} M$. We begin with a lemma, which will be referred to throughout the book, that collects many basic properties of essential extensions.

Lemma 1.1. *Let M denote a module.*

(1) If $K \subseteq N \subseteq M$ then $K \subseteq^{ess} M$ if and only if $K \subseteq^{ess} N$ and $N \subseteq^{ess} M$.
(2) If $K \subseteq^{ess} N \subseteq M$ and $K' \subseteq^{ess} N' \subseteq M$ then $K \cap K' \subseteq^{ess} N \cap N'$.
(3) If $\alpha : M \to N$ is R-linear and $K \subseteq^{ess} N$, then $\alpha^{-1}(K) \subseteq^{ess} M$, where $\alpha^{-1}(K) = \{m \in M \mid \alpha(m) \in K\}$.
(4) Let $M = \oplus_{i \in I} M_i$ be a direct sum where $M_i \subseteq M$ for each i, and let $K_i \subseteq M_i$ for each i. Then $\oplus_{i \in I} K_i \subseteq^{ess} M$ if and only if $K_i \subseteq^{ess} M_i$ for each i.

Proof. (1) and (2). These are routine verifications.

(3). Let $0 \neq X \subseteq M$; we must show that $X \cap \alpha^{-1}(K) \neq 0$. This is clear if $\alpha(X) = 0$ since then $X \subseteq \alpha^{-1}(K)$. Otherwise, $\alpha(X) \cap K \neq 0$ by hypothesis, say $0 \neq \alpha(x) \in K$, $x \in X$. Then $0 \neq x \in X \cap \alpha^{-1}(K)$.

(4). Write $K = \oplus_{i \in I} K_i$, and assume that $K_i \subseteq^{ess} M_i$ for each i. Then $K \subseteq^{ess} M$ if and only if $mR \cap K \neq 0$ for each $0 \neq m \in M$. Since m lies in a finite direct sum of the M_i, it suffices to prove (4) when I is finite, and hence (by induction) when $|I| = 2$. Let $\pi_1 : M_1 \oplus M_2 \to M_1$ be the projection with $ker(\pi_1) = M_2$. Then $K_1 \oplus M_2 = \pi_1^{-1}(K_1) \subseteq^{ess} M_1 \oplus M_2$ by (3). Similarly, $M_1 \oplus K_2 \subseteq^{ess} M_1 \oplus M_2$, and (4) follows from (2) because $K_1 \oplus K_2 = (K_1 \oplus M_2) \cap (M_1 \oplus K_2)$. □

This book is concerned with injective modules and their generalizations, and the main properties of these modules are derived in this section. A module E_R is called *injective* if whenever $0 \to N \xrightarrow{\alpha} M$ is R-monic, every R-linear map $\beta : N \to E$ factors in the form $\beta = \gamma \circ \alpha$ for some R-linear map

$$\begin{array}{ccc} 0 \to & N & \xrightarrow{\alpha} M \\ & \beta\downarrow & \overset{\gamma}{\swarrow} \\ & E & \end{array}$$

$\gamma : M \to E$. These modules admit a characterization that we will use repeatedly in the following.

Lemma 1.2. *A module E is injective if and only if, whenever $K \subseteq M$, every R-linear map $\beta : K \to E$ extends to an R-linear map $\gamma : M \to E$.*

Proof. The condition clearly holds if E is injective. Conversely, if $N \xrightarrow{\alpha} M$ is R-monic, the map $\alpha' : \alpha(N) \to N$ is well defined by $\alpha'(\alpha(n)) = n$ for $n \in N$. Then, given $\beta : N \to E$, the map $\beta \circ \alpha' : \alpha(N) \to E$ extends to $\gamma : M \to E$ by hypothesis, and one checks that $\gamma \circ \alpha = \beta$. □

Corollary 1.3. *If $E = \Pi_i E_i$ is a direct product of modules, then E is injective if and only if each E_i is injective.*

Proof. Let $E_i \xrightarrow{\sigma_i} E \xrightarrow{\pi_j} E_j$ be the canonical maps. If E is injective, and if $K \subseteq M$ and $\beta : K \to E_i$ are given, there exists $\gamma : M \to E$ such that $\gamma = \sigma_i \circ \beta$ on K. Then $\pi_i \circ \gamma : M \to E_i$ extends β, proving that E_i is injective by Lemma 1.2. Conversely, if each E_i is injective, let $\alpha : K \to E$, where $K \subseteq M$. For each i, there exists $\gamma_i : M \to E_i$ extending $\pi_i \circ \alpha$. If $\gamma : M \to E$ is defined by $\gamma(m) = \langle \gamma_i(m) \rangle$ for each $m \in M$, then γ extends α because $x = \langle \pi_i(x) \rangle$ for each $x \in M$. It follows that E is injective by Lemma 1.2. □

Surprisingly, to prove that a module E is injective, it is enough to verify the condition in Lemma 1.2 when $M = R$.

Lemma 1.4 (Baer Criterion). *A right R-module E is injective if and only if, whenever $T \subseteq R$ is a right ideal, every map $\gamma : T \to E$ extends to $R \to E$, that is, $\gamma = c\cdot$ is multiplication by an element $c \in E$.*

Proof. The condition is clearly necessary. To prove sufficiency, let $K \subseteq M$ be modules and let $\beta : K \to E$. In this case, let $\mathcal{F}$ denote the set of pairs (K', β') such that $K \subseteq K' \subseteq M$ and $\beta' : K' \to E$ extends β. By Zorn's lemma, let (K'', β'') be a maximal member of $\mathcal{F}$. We must show that $K'' = M$. If not, let $m \in M - K''$, let $T = \{r \in R \mid mr \in K''\}$ – a right ideal, and define $\lambda : T \to E$ by $\lambda(r) = \beta''(mr)$. By hypothesis there exists $\hat{\lambda} : R \to E$ extending λ, and we use it to define $\hat{\beta} : K'' + mR \to E$ by $\hat{\beta}(y + mr) = \beta''(y) + \hat{\lambda}(r)$, where $y \in K''$ and $r \in R$. This is well defined because $y + mr = 0$ implies that $mr \in K''$ and so $\hat{\lambda}(r) = \lambda(r) = \beta''(mr) = \beta''(-y) = -\beta''(y)$. Since $\hat{\beta}$ is R-linear and extends β'' this contradicts the maximality of (K'', β'') in $\mathcal{F}$. □

It is a routine matter to show that an (additive) abelian group X is injective as a $\mathbb{Z}$-module if and only if it is *divisible*, that is, $nX = X$ for any $0 \neq n \in \mathbb{Z}$. Examples include $\mathbb{Q}$ and the Prüfer group $\mathbb{Z}_{p^\infty}$ for any prime p. Divisible groups can be used to construct injective modules over any ring. The second part of the next lemma was discovered by Baer in 1940.

Lemma 1.5. *Let R be a ring. Then the following hold:*

(1) If Q is a divisible group then $E_R = hom_{\mathbb{Z}}(R, Q)$ is an injective right R-module.
*(2) (**Baer**) Every module M_R embeds in an injective right module.*

Proof. (1). If $\lambda \in E$ and $a \in R$, E becomes a right R-module via $(\lambda \cdot a)(r) = \lambda(ar)$ for all $r \in R$. Now let $\gamma : T \to E_R$ be R-linear, where T is a right ideal of R. By Lemma 1.4 we must extend γ to $R_R \to E_R$. Define $\theta : T \to Q$ by $\theta(t) = [\gamma(t)](1)$. Then θ is a $\mathbb{Z}$-morphism; so, since ${}_{\mathbb{Z}}Q$ is injective, let $\hat{\theta} : R \to Q$ be a $\mathbb{Z}$-morphism extending θ. Since $\hat{\theta} \in E$, define $\hat{\gamma} : R \to E$ by $\hat{\gamma}(a) = \hat{\theta} \cdot a$ for all $a \in R$. One verifies that $\hat{\gamma}$ is R-linear, and we claim that it extends γ; that is, $\hat{\gamma}(t) = \gamma(t)$ for all $t \in T$. If $r \in R$, we have

$$[\hat{\gamma}(t)](r) = [\hat{\theta} \cdot t](r) = \hat{\theta}(tr) = \theta(tr) = [\gamma(tr)](1) = [\gamma(t) \cdot r](1) = [\gamma(t)](r)$$

because γ is R-linear and $\gamma(t) \in E$. Hence $\hat{\gamma}(t) = \gamma(t)$, as required.

(2). Given M_R, let $\varphi : \mathbb{Z}^{(I)} \to M$ be $\mathbb{Z}$-epic for some set I, so that ${}_{\mathbb{Z}}M \cong \mathbb{Z}^{(I)}/K \subseteq \mathbb{Q}^I/K$, where $K = ker(\varphi)$. Write $Q = \mathbb{Q}^I/K$ and note that Q is divisible. Since $M_R \cong hom(R_R, M_R)$ via $m \mapsto m\cdot$, we get

$$M_R \cong hom_R(R_R, M_R) \subseteq hom_{\mathbb{Z}}(R, M) \hookrightarrow hom_{\mathbb{Z}}(R, Q).$$

Since $E_R = hom_{\mathbb{Z}}(R, Q)$ is injective by (1), this proves (2). □

Corollary 1.6. *A module E is injective if and only if every monomorphism $\sigma : E \to M$ splits, that is, $\sigma(E) \subseteq^{\oplus} M$.*

Proof. If $\sigma : E \to M$ is monic there exists $\gamma : M \to E$ such that $\gamma \circ \sigma = 1_E$. Then $M = \sigma(E) \oplus ker(\gamma)$. The converse is clear from Lemma 1.5 because direct summands of injective modules are injective. □

Before proceeding, we need another basic property of essential extensions. If K is a submodule of a module M, it is a routine application of Zorn's lemma to see that there exist submodules C of M maximal with respect to $K \cap C = 0$. Such a submodule C is called a *complement*[1] of K in M. Thus $K \subseteq^{ess} M$ if and only if 0 is a complement of K.

[1] It is sometimes called an *intersection complement*, or *relative complement*.

Lemma 1.7 (Essential Lemma). *Let $K \subseteq M$ be modules. If C is any complement of K in M then the following hold:*

(1) $K \oplus C \subseteq^{ess} M$.
(2) $(K \oplus C)/C \subseteq^{ess} M/C$.

Proof. (1). Let X be a nonzero submodule of M; we must show that $X \cap (K \oplus C) \neq 0$. This is clear if $X \subseteq C$. Otherwise the maximality of C shows that $K \cap (X + C) \neq 0$, say $0 \neq k = x + c$ with the obvious notation. Hence $x \in X \cap (K \oplus C)$, and $x \neq 0$ because $K \cap C = 0$.

(2). Let $Y/C \cap (K \oplus C)/C = 0$. If $Y \neq C$ then $Y \cap K \neq 0$ by the choice of C, say $0 \neq a \in Y \cap K$. Then $a + C \in Y/C \cap (K \oplus C)/C = 0$ so $a \in C$. But then $0 \neq a \in C \cap K = 0$, which is a contradiction. □

Given any module M, an R-monomorphism $M \xrightarrow{\sigma} E$ is called an *injective hull* (*injective envelope*) of M if E is injective and $\sigma(M) \subseteq^{ess} E$. The following result is a famous theorem that traces back to Baer, to Eckmann and Schopf, and to Shoda.

Theorem 1.8 (Baer/Eckmann–Schopf/Shoda). *Let M_R be a module.*

(1) M has an injective hull.
(2) If $M \xrightarrow{\sigma_1} E_1$ and $M \xrightarrow{\sigma_2} E_2$ are two injective hulls there exists an isomorphism $\tau : E_1 \to E_2$ such that $\sigma_2 = \tau \circ \sigma_1$.

Proof. (1). By Lemma 1.5 let $M \subseteq Q_R$ where Q_R is injective, and, by Zorn's lemma, let E be maximal such that $M \subseteq^{ess} E \subseteq Q$. Then let $C \subseteq Q$ be maximal such that $E \cap C = 0$; it suffices to show that $E \oplus C = Q$ (so E is injective). By Lemma 1.7 we have $E \cong (E \oplus C)/C \subseteq^{ess} Q/C$. Define $\sigma : (E \oplus C)/C \to Q$ by $\sigma(x + C) = x$ if $x \in E$. Since Q is injective, σ extends to $\hat{\sigma} : Q/C \to Q$. Then $\hat{\sigma}$ is monic because $ker(\hat{\sigma}) \cap (E \oplus C)/C = 0$, and so $im(\sigma) = \hat{\sigma}((E \oplus C)/C) \subseteq^{ess} \hat{\sigma}(Q/C)$. Since $M \subseteq^{ess} E = im(\sigma)$ it follows that $E \subseteq^{ess} \hat{\sigma}(Q/C)$, and so $E = \hat{\sigma}(Q/C)$ by the maximality of E. But then $\hat{\sigma}(Q/C) = E = \hat{\sigma}((E \oplus C)/C)$ and we conclude that $Q = E \oplus C$ because $\hat{\sigma}$ is monic. This is what we wanted.

(2). The given map τ exists because E_2 injective. Moreover, τ is monic because $ker(\tau) \cap \sigma_1(M) = 0$ (since σ_2 is monic) and $\sigma_1(M) \subseteq^{ess} E_1$. Hence $\tau(E_1) \subseteq^{\oplus} E_2$ by Corollary 1.6. But $\tau(E_1) \subseteq^{ess} E_2$ because $\sigma_2(M) = \tau\sigma_1(M) \subseteq \tau(E_1)$ and $\sigma_2(M) \subseteq^{ess} E_2$ by hypothesis. It follows that τ is onto and so is an isomorphism. □

Hence we are entitled to speak of *the* injective hull of a module M and to denote it by $E(M)$. We will usually assume that $M \subseteq E(M)$; so, for example, we have $E(\mathbb{Z}) = \mathbb{Q}$ and $E(\mathbb{Z}_{p^n}) = \mathbb{Z}_{p^\infty}$ for any prime p and $n \geq 2$. The assumption that $M \subseteq E(M)$ is justified by the following result.

Lemma 1.9. *Let $\sigma : M \to E(M)$ be an injective hull of the module M. If $M \subseteq G$, where G is any injective module, there exists a copy $E \cong E(M)$ inside G such that $M \subseteq^{ess} E \subseteq^{\oplus} G$.*

Proof. As G is injective, there exists $\tau : E(M) \to G$ such that $m = \tau\sigma(m)$ for every $m \in M$. Since $ker(\tau) \cap \sigma(M) = 0$ it follows that τ is monic, and we are done by Corollary 1.6 with $E = \tau[E(M)]$. □

Lemma 1.9 will be used frequently in the following, usually without comment. In particular, let $M = \oplus_{i=1}^n M_i$ be a direct sum of modules, and let $M \subseteq E(M)$. By Lemma 1.9 we can choose a copy of $E(M_i)$ such that $M_i \subseteq^{ess} E(M_i) \subseteq E(M)$ for each i. One verifies that $E(M_1) \cap E(M_2) = 0$, so (by Lemma 1.1) $M_1 \oplus M_2 \subseteq^{ess} E(M_1) \oplus E(M_2)$. Continuing inductively, we conclude that $\Sigma_{i=1}^n E(M_i)$ is direct and that $M = \oplus_{i=1}^n M_i \subseteq^{ess} \oplus_{i=1}^n E(M_i)$. Since $\oplus_{i=1}^n E(M_i)$ is injective (Corollary 1.3) we have proved the following:

Proposition 1.10. *If $M = \oplus_{i=1}^n M_i$ is a finite direct sum of modules then $E(\oplus_{i=1}^n M_i) = \oplus_{i=1}^n E(M_i)$.*

1.2. Relative Injectivity

Let M and G denote right R-modules. We say that G is M-injective if, for any submodule $X \subseteq M$, every R-linear map $\beta : X \to G$ can be extended to an R-linear map $\hat{\beta} : M \to G$, equivalently (see the proof of Lemma 1.2) if, for every

$$\begin{array}{ccc} X & \hookrightarrow & M \\ \beta\downarrow & \overset{\hat{\beta}}{\swarrow} & \\ G & & \end{array}$$

monomorphism $\sigma : X \to M$ there exists $\lambda : M \to G$ such that $\beta = \lambda \circ \sigma$. Thus G is injective if and only if it is M-injective for every module M, equivalently (by the Baer criterion) if G is R-injective. The proof of Corollary 1.3 gives

Lemma 1.11. *Let $G = \Pi_{i \in I} G_i$ and M be modules. Then G is M-injective if and only if G_i is M-injective for each $i \in I$.*

Lemma 1.12. *If G is M-injective and $N \subseteq M$ then G is both N-injective and (M/N)-injective.*

Proof. Given $X \xrightarrow{\beta} G$, where $X \subseteq N$, extend β to $\hat{\beta} : M \to G$ by hypothesis. Then the restriction $\hat{\beta}_{|N} : N \to G$ extends β, so G is N-injective. Now let $\alpha : X/N \to G$, $N \subseteq X \subseteq M$, and let $\pi : X \to X/N$ be the coset map. Then $\alpha \circ \pi : X \to G$ extends to $\lambda : M \to G$ by hypothesis. Hence $\hat{\alpha} : M/N \to G$ is well defined by $\hat{\alpha}(m + N) = \lambda(m)$, and $\hat{\alpha}$ extends α. This shows that G is (M/N)-injective. □

Note that if G is both N- and (M/N)-injective it does not follow that G is M-injective. Indeed, there is a monomorphism $\mathbb{Z}_p \xrightarrow{\sigma} \mathbb{Z}_{p^2}$ of abelian groups, given by $\sigma(n + p\mathbb{Z}) = pn + p^2\mathbb{Z}$ for all $n \in \mathbb{Z}$. Let $G = \mathbb{Z}_p$ and $N = im(\sigma)$. Then G is both N- and $(\mathbb{Z}_{p^2}/N)$-injective (because N and $\mathbb{Z}_{p^2}/N$ are simple), but G is not $\mathbb{Z}_{p^2}$-injective because any map $\lambda : \mathbb{Z}_{p^2} \to \mathbb{Z}_p$ satisfies $\lambda(N) = 0$. However, we do have

Lemma 1.13 (Azumaya's Lemma). *If G and $M = M_1 \oplus \cdots \oplus M_n$ are modules, then G is M-injective if and only if G is M_i-injective for each $i = 1, 2, \ldots, n$.*

Proof. If G is M-injective, then G is M_i-injective for each i by Lemma 1.12. Conversely, if G is M_i-injective for each i, let $\beta : X \to G$ be R-linear, where $X \subseteq M$. As in the proof of Lemma 1.4, let (C, β^*) be maximal such that $X \subseteq C \subseteq M$ and $\beta^* : C \to G$ extends β. We show $C = M$ by proving that $M_i \subseteq C$ for each i. By hypothesis there exists $\alpha_i : M_i \to G$ such that $\alpha_i = \beta^*$ on $M_i \cap C$. Construct $\beta_i : M_i + C \to G$ by $\beta_i(m_i + c) = \alpha_i(m_i) + \beta^*(c)$ for all $m_i \in M_i$ and $c \in C$. Then β_i is well defined because $\alpha_i = \beta^*$ on $M_i \cap C$, and β_i extends β because $X \subseteq C$ and β^* extends β. Hence $M_i + C = C$ by the maximality of (C, β^*), so $M_i \subseteq C$, as required. □

It is not surprising that there is a characterization of when G is M-injective in terms of the injective hulls $E(G)$ and $E(M)$.

Lemma 1.14. *A module G is M-injective if and only if $\lambda(M) \subseteq G$ for all R-linear maps $\lambda : E(M) \to E(G)$.*

Proof. If the condition holds, let $\beta : X \to G$ be R-linear, where $X \subseteq M$. Since $E(G)$ is injective there exists $\hat{\beta} : E(M) \to E(G)$ extending β. But $\hat{\beta}(M) \subseteq G$ by hypothesis, so the restriction $\hat{\beta}_{|M} : M \to G$ extends β.

Conversely, assume that G is M-injective, and let $\lambda : E(M) \to E(G)$ be R-linear. We must show that $\lambda(M) \subseteq G$. If $X = \{x \in M \mid \lambda(x) \in G\}$ then the restriction $\lambda_{|X} : X \to G$ extends to $\mu : M \to G$. Hence it suffices to show that $(\lambda - \mu)(M) = 0$. Since $G \subseteq^{ess} E(G)$, it is enough to show that $G \cap (\lambda - \mu)(M) = 0$. But if $g = (\lambda - \mu)(m)$, where $g \in G$ and $m \in M$, then $\lambda(m) = \mu(m) + g \in G$, so $m \in X$. This means that $\lambda(m) = \mu(m)$ by the definition of μ. Hence $g = \lambda(m) - \mu(m) = 0$, as required. □

A module M is called *quasi-injective* if it is M-injective, that is, if every map $\beta : X \to M$, where X is a submodule of M, extends to an endomorphism of M. Clearly every injective or semisimple module is quasi-injective, but the converse is false (for example, $\mathbb{Z}_4$ is quasi-injective as a $\mathbb{Z}$-module, as we shall see).

Lemma 1.14 leads to an important characterization of quasi-injective modules. We say that a submodule $K \subseteq M$ is *fully invariant* in M if $\lambda(K) \subseteq K$ for every $\lambda \in end(M)$. Then taking $G = M$ in Lemma 1.14 gives immediately

Lemma 1.15 (Johnson–Wong Lemma). *A module is quasi-injective if and only if M is fully invariant in its injective hull $E(M)$.*

Thus, for example, $\mathbb{Z}_{p^n}$ is quasi-injective as a $\mathbb{Z}$-module for any prime p because it is fully invariant in its injective hull $\mathbb{Z}_{p^\infty}$.

Corollary 1.16. *Let M be a quasi-injective module. If $E(M) = \oplus_{i \in I} K_i$, then $M = \oplus_{i \in I}(M \cap K_i)$.*

Proof. Let $m = \Sigma_{i=1}^n k_i \in M$, where each $k_i \in K_i$. If $\pi_i : E(M) \to E(M)$ is the projection onto K_i, then $k_i = \pi_i(m) \in \pi_i(M) \subseteq M$ by Lemma 1.15, so $k_i \in M \cap K_i$. Hence $M \subseteq \oplus_{i \in I}(M \cap K_i)$; the other inclusion is clear. □

If p is a prime, the $\mathbb{Z}$-module $\mathbb{Q} \oplus \mathbb{Z}_p$ is not quasi-injective even though $\mathbb{Q}$ is injective and $\mathbb{Z}_p$ is simple. (The coset map $\mathbb{Z} \to \mathbb{Z}_p$ does not extend to $\mathbb{Q} \oplus \mathbb{Z}_p \to \mathbb{Z}_p$ because there is no nonzero map $\mathbb{Q} \to \mathbb{Z}_p$.) Hence the direct sum of two quasi-injective modules need not be quasi-injective. However, we do have the following lemma:

Lemma 1.17. *If M is quasi-injective so also is every direct summand N.*

Proof. If $M = N \oplus N'$ and $\beta : X \to N$ is R-linear, where $X \subseteq N$, then β extends to $\hat{\beta} : M \to M$ by hypothesis. If $\pi : M \to N$ is the projection with kernel N', then $\lambda = (\pi \circ \hat{\beta})_{|N}$ is in $end(N)$ and extends β. □

The next result uses Lemma 1.15 to identify when a finite direct sum of quasi-injective modules is again of the same type.

Proposition 1.18. *Let $M_1, \dots, M_n$ be modules and write $E_i = E(M_i) \supseteq M_i$ for each i. The following are equivalent:*

(1) $M_1 \oplus \cdots \oplus M_n$ is quasi-injective.
(2) $\lambda(M_i) \subseteq M_j$ for all R-linear maps $\lambda : E_i \to E_j$.

Proof. Let $M_j \xrightarrow{\sigma_j} \oplus_k M_k \xrightarrow{\pi_i} M_i$ denote the canonical maps, and write $E = E(\oplus_k M_k) = \oplus_k E_k$.

(1)⇒(2). Given (1) and $\lambda : E_i \to E_j$, let $m_i \in M_i$. We have $\pi_j \circ \sigma_j = 1_{E_j}$ for each j, so $\lambda(m_i) = (\pi_j \sigma_j \lambda \pi_i \sigma_i)(m_i) = \pi_j(\sigma_j \lambda \pi_i)(\sigma_i m_i) \in M_j$ because $(\sigma_j \lambda \pi_i)(\oplus_k M_k) \subseteq \oplus_k M_k$ by (1) and Lemma 1.15.

(2)⇒(1). Given $\lambda : \oplus_k E_k \to \oplus_k E_k$, we must show (by Lemma 1.15) that $\lambda(\oplus_k M_k) \subseteq \oplus_k M_k$. Let $\bar{m} = m_1 + \cdots + m_n \in \oplus_k M_k$. Since $\Sigma_k \sigma_k \pi_k = 1_E$, we compute

$$\pi_j \lambda(\bar{m}) = \pi_j \lambda(\Sigma_k \sigma_k \pi_k \bar{m}) = \Sigma_k(\pi_j \lambda \sigma_k)(\pi_k \bar{m}) = \Sigma_k(\pi_j \lambda \sigma_k)(m_k) \in \oplus_k M_k$$

because $(\pi_j \lambda \sigma_k)(M_k) \subseteq M_j$ for all j and k by (2). □

Thus, for example, $\mathbb{Z}_n$ is quasi-injective as a $\mathbb{Z}$-module for each $n \in \mathbb{Z}$. In fact, $\mathbb{Z}_n = \mathbb{Z}_{p_1^{n_1}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{n_k}}$ for distinct primes p_i, each $\mathbb{Z}_{p_i^{n_i}}$ is quasi-injective, and $hom_{\mathbb{Z}}(\mathbb{Z}_{p^\infty}, \mathbb{Z}_{q^\infty}) = 0$ if p and q are distinct primes.

Corollary 1.19. *A module M is quasi-injective if and only if M^n is quasi-injective.*

1.3. Continuous Modules

In his work on continuous rings, Utumi identified three conditions on a ring that are satisfied if the ring is self-injective. The analogs of these conditions for a module M are as follows:

(1) M satisfies the C1-*condition* if every submodule of M is essential in a direct summand of M.[2] (Note that we regard the zero submodule as essential in itself.)
(2) M satisfies the C2-*condition* if every submodule that is isomorphic to a direct summand of M is itself a direct summand.

[2] This condition is also referred to as the CS-*condition* because it is equivalent to the requirement that every complement submodule is a direct summand (complement submodules are also called *closed* submodules). We return to this topic in the following section.

(3) M satisfies the C3-*condition* if, whenever N and K are submodules of M with $N \subseteq^{\oplus} M$, $K \subseteq^{\oplus} M$, and $N \cap K = 0$, then $N \oplus K \subseteq^{\oplus} M$.

A ring R is called a right C1 *ring* (respectively C2 *ring*, C3 *ring*) if the module R_R has the corresponding property.

If M is an indecomposable module then M is a C3 module; M is a C1 module if and only if it is *uniform* (that is $X \cap Y \neq 0$ for all submodules $X \neq 0$ and $Y \neq 0$) and M is a C2 module if and only if monomorphisms in $end(M)$ are isomorphisms. The $\mathbb{Z}$-modules $\mathbb{Z}_2$ and $\mathbb{Z}_8$ each satisfy the C1-, C2- and C3-conditions, but their direct sum $N = \mathbb{Z}_2 \oplus \mathbb{Z}_8$ is not a C1 module because, writing $S = \mathbb{Z}_2 \oplus 0$ and $K = \mathbb{Z}(1 + 2\mathbb{Z}, 2 + 8\mathbb{Z})$, we see that K is contained in only two direct summands N and $S \oplus K$ and is essential in neither. Moreover, N is not a C2 module because the non-summand $0 \oplus \mathbb{Z}(4 + 8\mathbb{Z})$ is isomorphic to the summand $\mathbb{Z}_2 \oplus 0$. Hence a direct sum of C1 modules, or C2 modules, may not inherit the same property.

As an abelian group, $\mathbb{Z}$ satisfies both the C1- and C3-conditions, but it is not a C2 module. However, if F is a field let $R = \begin{bmatrix} F & V \\ 0 & F \end{bmatrix}$, where $V = F \oplus F$. If $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ then $eR = \begin{bmatrix} F & V \\ 0 & 0 \end{bmatrix}$ is indecomposable (in fact $eRe \cong F$) and is a C2 module because monomorphisms are epic, but it is not a C1 module because it is not uniform.

Example 1.20. Let $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$, where F is a field. Then R is a right and left C1 ring, but neither a left nor right C2 ring.

Proof. We have $J = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix} \cong e_{12}R$ (where e_{ij} is the matrix unit), so R is not right C2 because J_R is not a direct summand of R_R. Similarly, R is not left C2. To see that R is right C1, let $T \neq 0$ be a right ideal. If $T \nsubseteq S_r = \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}$ then $T = e_{11}R$ or $T = R$, so T is a summand. If $T = S_r$ then $T \subseteq^{ess} R_R$ because R is right artinian. So we may assume that $dim_F(T) = 1$, say $T = xR$, $x \in S_r$. If $x^2 = x \neq 0$ we are done. Otherwise $x \in J$, so $T = J$ and one verifies that $T \subseteq^{ess} e_{11}R = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}$. Hence R is right C1; similarly R is right C2. □

Lemma 1.21. *The C2-condition implies the C3-condition.*

Proof. Let $N \subseteq^{\oplus} M$ and $K \subseteq^{\oplus} M$ satisfy $N \cap K = 0$; we must show that $N \oplus K \subseteq^{\oplus} M$. Write $M = N \oplus N'$, and let $\pi : M \to N'$ be the projection with $ker(\pi) = N$. If $k \in K$ and $k = n + n'$, $n \in N$, $n' \in N'$, then $\pi(k) = n'$ and it follows that $N \oplus K = N \oplus \pi(K)$. Hence we show that $N \oplus \pi(K) \subseteq^{\oplus} M$. Since $\pi_{|K} : K \to M$ is monic we have $\pi(K) \subseteq^{\oplus} M$ by the C2-condition. Since $\pi(K) \subseteq N'$, it follows that $N' = \pi(K) \oplus W$ for some submodule W and hence that $M = N \oplus \pi(K) \oplus W$. Thus M satisfies the C3-condition. □

A module is called *continuous* if it satisfies both the C1- and C2-conditions, and a module is called *quasi-continuous* if it satisfies the C1- and C3-conditions, and R is called a right *continuous ring* (right *quasi-continuous ring*) if R_R has the corresponding property. As the terminology suggests, every continuous module is quasi-continuous (by Lemma 1.21). Clearly every injective or semisimple module is continuous; in fact:

Proposition 1.22. *Every quasi-injective module is continuous.*

Proof. Let M be quasi-injective. If $N \subseteq M$ then $E(M)$ contains a copy of $E(N) = E$, and $E(M) = E \oplus G$ for some submodule G because E is injective. But then Corollary 1.16 shows that $M = (M \cap E) \oplus (M \cap G)$. Moreover, $N \subseteq^{ess} E$, so $N \subseteq^{ess} (M \cap E)$. This shows that M has the $C1$-property.

Now suppose that $N \cong P \subseteq^{\oplus} M$. Since M is M-injective, it follows from Lemma 1.11 that P is also M-injective and hence that N is M-injective. But then the identity map $1_N : N \to N$ extends to $\lambda : M \to N$, and it follows that $M = N \oplus ker(\lambda)$. This proves C2. □

The following lemma is a useful connection between essential submodules and singular modules.

Lemma 1.23. *If* $K \subseteq^{ess} M$ *are modules then* M/K *is singular, that is,* $Z(M/K) = M/K$.

Proof. If $K \subseteq^{ess} M$ and $m \in M$, we must show that $\mathtt{r}_R(m + K) \subseteq^{ess} R_R$, that is, $bR \cap \mathtt{r}_R(m + K) \neq 0$ for every $0 \neq b \in R$. This is clear if $mb = 0$. Otherwise, we have $mbR \cap K \neq 0$ by hypothesis, say $0 \neq mba \in K$, $a \in R$. But then $0 \neq ba \in bR \cap \mathtt{r}_R(m + K)$, as required. □

We can now prove two important results about endomorphism rings. A ring R is called *semiregular*[3] if R/J is (von Neumann) regular and idempotents lift modulo J, equivalently (by Lemma B.40 in Appendix B) if, for each $a \in R$ there exists $e^2 = e \in aR$ such that $(1 - e)a \in J$. We are going to prove that the endomorphism ring S of a continuous module M_R is semiregular and $J(S) = \{\alpha \in S \mid ker(\alpha) \subseteq^{ess} M\}$. We will need the following lemma.

Lemma 1.24. *Given* M_R, *write* $S = end(M)$ *and* $\bar{S} = S/J(S)$, *and assume* S *is semiregular and* $J(S) = \{\alpha \in S \mid ker(\alpha) \subseteq^{ess} M\}$.

(1) If $\pi^2 = \pi$ *and* $\tau^2 = \tau$ *in* S *satisfy* $\bar{\pi}\bar{S} \cap \bar{\tau}\bar{S} = 0$ *then* $\pi M \cap \tau M = 0$.
(2) If M *satisfies the C3-condition and* $\Sigma_{i \in I}\bar{\pi}_i\bar{S}$ *is direct in* $\bar{S}$, *where* $\pi_i^2 = \pi_i \in S$ *for each* i, *then* $\Sigma_{i \in I}\pi_i M$ *is direct in* M.

[3] These rings are also called *F-semiperfect* in the literature.

(3) *If M is quasi-continuous and $\Sigma_{i \in I}\pi_i M$ is direct in M, where $\pi_i^2 = \pi_i \in S$ for each i, then $\Sigma_{i \in I}\bar{\pi}_i \bar{S}$ is direct in $\bar{S}$.*

Proof. (1). We begin with a simplifying adjustment.

Claim 1. We may assume that $\bar{\tau}\bar{\pi} = 0$.

Proof. As $\bar{S}$ is regular, let $\bar{\pi}\bar{S} \oplus \bar{\tau}\bar{S} \oplus T = \bar{S}$, with T a right ideal of $\bar{S}$. Let $\bar{\eta}^2 = \bar{\eta}$ be such that $\bar{\tau}\bar{S} = \bar{\eta}\bar{S}$ and $\bar{\pi}\bar{S} \oplus T = (1 - \bar{\eta})\bar{S}$. By hypothesis, we may assume that $\eta^2 = \eta$ in S. Note that $\bar{\eta}\bar{\pi} = 0$. Then $\gamma = \tau + \tau\eta(1 - \tau)$ satisfies $\gamma^2 = \gamma$, $\gamma\tau = \tau$, and $\tau\gamma = \gamma$, so $\tau M = \gamma M$ and $\bar{\tau}\bar{S} = \bar{\gamma}\bar{S}$. Moreover, $\bar{\gamma} = \bar{\eta}$ because $\bar{\tau}\bar{\eta} = \bar{\eta}$ and $\bar{\eta}\bar{\tau} = \bar{\tau}$. Hence $\bar{\gamma}\bar{\pi} = \bar{\eta}\bar{\pi} = 0$, so replacing τ by γ proves Claim 1.

By Claim 1 we have $\tau\pi \in J(S)$, so writing $K = ker(\tau\pi)$, we have $K \subseteq^{ess} M$ by hypothesis. It follows that $\pi K \subseteq^{ess} \pi M$ (if $0 \neq X \subseteq \pi M$ then $\pi x = x$ for each $x \in X$, so $0 \neq X \cap K \subseteq X \cap \pi K$). However, $\tau\pi K = 0$, so $\pi K \subseteq ker(\tau)$, whence $\pi K \cap \tau M = 0$. This in turn implies that $\pi M \cap \tau M = 0$ because $\pi K \subseteq^{ess} \pi M$.

(2). It is enough to show that $\Sigma_{i \in F}\pi_i M$ is direct for any finite subset $F \subseteq I$. Write $F = \{1, \ldots, n\}$ and proceed by induction on n. If $n = 1$ there is nothing to prove, and if $n = 2$ then (2) follows from (1). Assume inductively that $\pi_1 M + \cdots + \pi_n M$ is a direct sum, $n \geq 1$. Then the C3-condition implies that $\pi_1 M \oplus \cdots \oplus \pi_n M = \pi M$ for some $\pi^2 = \pi \in S$.

Claim 2. $\pi S = \pi_1 S + \cdots + \pi_n S$.

Proof. We have $\pi\pi_i = \pi_i$ for each i (because $\pi_i M \subseteq \pi M$), so $\Sigma_{i=1}^n \pi_i S \subseteq \pi S$. For each $i = 1, \ldots, n$, let $\rho_i : \pi M \to \pi_i M$ be the projection, so that $\pi\rho_i = \rho_i$ for each i and $\pi = \Sigma_{i=1}^n \rho_i\pi = \Sigma_{i=1}^n \tau_i$, where we define $\tau_i = \rho_i\pi$ for each i. Then $\pi\tau_i = \tau_i$ for each i, and so $\pi S = \Sigma_{i=1}^n \tau_i S = \Sigma_{i=1}^n \pi_i\tau_i S \subseteq \Sigma_{i=1}^n \pi_i S$. This proves Claim 2.

By Claim 2 we have $\bar{\pi}\bar{S} = \bar{\pi}_1\bar{S} \oplus \cdots \oplus \bar{\pi}_n\bar{S}$, so $\bar{\pi}\bar{S} \cap \bar{\pi}_{n+1}\bar{S} = 0$. But then (1) implies that $\pi M \cap \pi_{n+1}M = 0$. Since $\pi_1 M \oplus \cdots \oplus \pi_n M = \pi M$, this shows that $\pi_1 M \oplus \cdots \oplus \pi_n M \oplus \pi_{n+1} M$ is a direct sum, as required.

(3). For each $i \in I$ let C_i be a closure of $\oplus_{j \neq i}\pi_j M$, so that $\oplus_{j \neq i}\pi_j M \subseteq^{ess} C_i$. It follows that $\pi_i M \cap C_i = 0$. But C_i is a direct summand of M by the C1-condition, so the C3-condition implies that $M = \pi_i M \oplus C_i \oplus N_i$ for some submodule $N_i \subseteq M$. So, for each $i \in I$, let $\tau_i^2 = \tau_i \in S$ satisfy $\tau_i M = \pi_i M$ and $ker(\tau_i) = C_i \oplus N_i$. Then $\pi_i\tau_i = \tau_i$ and $\tau_i\pi_i = \pi_i$, and so $\tau_i S = \pi_i S$. Furthermore, $\tau_i\pi_j = 0$ for all $j \neq i$ because $\pi_j M \subseteq C_i \subseteq ker(\tau_i)$. But then

$\tau_i\tau_j = \tau_i(\pi_j\tau_j) = (\tau_i\pi_j)\tau_j = 0$ whenever $i \neq j$, so the τ_i are orthogonal. Thus $\Sigma_i\bar{\pi}_i\bar{S} = \Sigma_i\bar{\tau}_i\bar{S}$ is direct in $\bar{S}$ because the $\bar{\tau}_i$ are also orthogonal. □

We can now prove an important result about the endomorphism ring of a continuous (or quasi-injective) module.

Theorem 1.25. *Let M_R be a continuous module with $S = end(M_R)$. Then:*

(1) S is semiregular and $J(S) = \{\alpha \in M \mid ker(\alpha) \subseteq^{ess} M\}$.
(2) $S/J(S)$ is right continuous.
(3) If M is actually quasi-injective, $S/J(S)$ is right self-injective.

Proof. (1). Write $\Delta = \{\alpha \in S \mid ker(\alpha) \subseteq^{ess} M\}$. It is a routine exercise to show that Δ is a left ideal of S; it is also a right ideal using (3) of Lemma 1.1. If $\alpha \in \Delta$ the fact that $ker(\alpha) \cap ker(1-\alpha) = 0$ means that $ker(1-\alpha) = 0$. Hence $(1-\alpha)M \cong M$, so, by C2, $(1-\alpha)M \subseteq^{\oplus} M$. But $ker(\alpha) \subseteq (1-\alpha)M$, so it follows that $(1-\alpha)M = M$. Hence $1-\alpha$ is a unit in S, and it follows that $\Delta \subseteq J(S)$.

Let $\alpha \in S$ and (by C1) let $ker(\alpha) \subseteq^{ess} P$ where $P \oplus Q = M$. Then $\alpha Q \cong Q$, so (by C2) let $\alpha Q \oplus W = M$. Then $\beta \in S$ is well defined by $\beta(\alpha q + w) = q$, $q \in Q$, $w \in W$. If $\pi^2 = \pi \in S$ satisfies $\pi M = Q$, then $\beta\alpha\pi = \pi$. Define $\tau = \alpha\pi\beta$. Then $\tau^2 = \tau \in \alpha S$ and $(1-\tau)\alpha = \alpha - \alpha\pi\beta\alpha$ is in Δ because $ker(\alpha - \alpha\pi\beta\alpha) \supseteq ker(\alpha) \oplus Q$ and $ker(\alpha) \oplus Q \subseteq^{ess} P \oplus Q = M$ (by Lemma 1.1). It follows that S/Δ is regular and hence that $J(S) \subseteq \Delta$. This proves that $J(S) = \Delta$ and so S is semiregular by Lemma B.40. This proves (1).

In preparation for the proof of (2) and (3), let T be a right ideal of $\bar{S}$ and, by Zorn's lemma, choose a family $\{\bar{\pi}_i\bar{S} \mid i \in I\}$ of nonzero, principal right ideals of $\bar{S}$ maximal such that $\bar{\pi}_i\bar{S} \subseteq T$ for each i and $\Sigma_i\bar{\pi}_i\bar{S}$ is direct. Since $\bar{S}$ is regular, we may assume that each $\bar{\pi}_i$ is an idempotent; since idempotents lift modulo $J(S)$ we may further assume that $\pi_i^2 = \pi_i$ in S. Then $\Sigma_i\pi_i M$ is direct by Lemma 1.24.

(2). Let $\oplus_i\pi_i M \subseteq^{ess} \pi M$, where $\pi^2 = \pi \in S$ (by C1). Since $\pi_i M \subseteq \pi M$ for each i, we have $\bar{\pi}_i\bar{S} \subseteq \bar{\pi}\bar{S}$, so $\oplus_i\bar{\pi}_i\bar{S} \subseteq \bar{\pi}\bar{S}$.

Claim. $\oplus_i\bar{\pi}_i\bar{S} \subseteq^{ess} \bar{\pi}\bar{S}$.

Proof. Suppose that $\bar{\eta}\bar{S} \cap (\oplus_i\bar{\pi}_i\bar{S}) = 0$, where $\bar{\eta} \in \bar{\pi}\bar{S}$. As before, we may assume that $\eta^2 = \eta$ in S. Thus $\eta M \cap (\oplus_i\pi_i M) = 0$ by Lemma 1.24. Since $\bar{\pi}\bar{\eta} = \bar{\eta}$, we have $(\pi\eta - \eta) \in J(S) = \Delta$ and so $(\pi\eta - \eta)K = 0$ for some $K \subseteq^{ess} M$. But this implies that $\eta K \subseteq \pi M$, and it follows that $\eta K = 0$ because $(\oplus_i\pi_i M) \subseteq^{ess} \pi M$. Hence $\eta \in J(S)$, so $\bar{\eta} = 0$. This proves the Claim.