QUASI-FROBENIUS RINGS

The study of quasi-Frobenius rings grew out of the theory of group representations in the 1940s and has produced an enormous body of results. This book makes no attempt to be encyclopedic but provides an elementary account of the basic facts about these rings at a level allowing researchers and graduate students to gain entry to the field. Many earlier results about self-injective rings are extended to the much wider class of mininjective rings; the methods used unify and simplify what is known in the area and so bring the reader up to current research. Sufficient background knowledge can be found in standard texts on noncommutative rings. However, appendices on Morita equivalence; on perfect, semiperfect, and semiregular rings; and on the Camps–Dicks theorem are included to make the book self-contained. After the basic results are established in Chapters 1 through 6, recent work is reviewed on three open problems in the field (the Faith conjecture, the FGF-conjecture, and the Faith–Menal conjecture). Some new results are provided and new and old methods for attacking these problems are outlined in an easily accessible format.

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158 Quasi-Frobenius Rings

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For Kathleen and Eman

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List of Symbols

$\mathbb{N}$	Set of natural numbers
$\mathbb{Z}$	Ring of integers
$\mathbb{Q}$	Field of rational numbers
$\mathbb{R}$	Field of real numbers
$\mathbb{C}$	Field of complex numbers
$\mathbb{Z}_n$	Ring of integers modulo n
$\mathbb{Z}_{(p)}$	Integers localized at the prime p
$\mathbb{Z}_{p^\infty}$	Prüfer group at the prime p
δ_{ij}	Kronecker delta
$\|X\|$	Cardinality of a set X
$X \subset Y$	$X \subseteq Y$ and $X \neq Y$, for sets X and Y
$E(M)$	Injective hull of the module M
$soc(M)$	Socle of the module M
$rad(M)$	Radical of the module M
$dim(M)$	Uniform (Goldie) dimension of the module M
$length(M)$	Composition length of the module M
$Z(M)$	Singular submodule of the module M
$char(R)$	Characteristic of the ring R
S_r, S_l	$soc(R_R)$, $soc(_RR)$
Z_r, Z_l	$Z(R_R)$, $Z(_RR)$
J, $J(R)$	Jacobson radical of the ring R
$\mathtt{r}(X)$, $\mathtt{l}(X)$	Left and right annihilators of the set X
$R[x]$	Polynomial ring over the ring R
$F(x)$	Ring of rational functions over the field F
$M_n(R)$	Ring of $n \times n$ matrices over the ring R
R^n, R_n	Row matrices, column matrices over the ring R
$end(M)$	Endomorphism ring of the module M
$K \subseteq^{ess} M$	K is an essential submodule of the module M

$K \subseteq^{max} M$	K is a maximal submodule of the module M
$K \subseteq^{sm} M$	K is a small submodule of the module M
$K \subseteq^{\oplus} M$	K is a direct summand of the module M
$c\cdot,\ \cdot c$	Left (right) multiplication map by the element c
$M^{(I)}$	The direct sum of $\lvert I\rvert$ copies of the module M
M^I	The direct product of $\lvert I\rvert$ copies of the module M
$lat(M)$	Lattice of submodules of the module M
M^*	Dual of the module M
$modR,\ Rmod$	Categories of right and left modules over the ring R
$V \otimes_R W,\ v \otimes w$	Tensor product of modules, elements

Preface

A ring R is called *quasi-Frobenius* if it is right or left self-injective and right or left artinian (all four combinations being equivalent). The study of these rings grew out of the theory of representations of a finite group as a group of matrices over a field – the corresponding group algebra is quasi-Frobenius. At the turn of the twentieth century G. Frobenius carried out fundamental work on representations of "hypercomplex systems" – finite dimensional algebras in modern terminology. This topic was revived in the late 1930s and early 1940s by Brauer, Nesbitt, Nakayama, and others in their study of "Frobenius algebras." Nakayama introduced quasi-Frobenius rings in 1939 and, in 1951 Ikeda characterized them as the left and right self-injective, left and right artinian rings. The subject is intimately related to duality, the duality from right to left modules induced by the hom functor, and, more importantly for us, the duality related to annihilators. The present extent of the theory is vast, and we make no attempt to be encyclopedic here. Instead we provide an elementary, self-contained account of the basic facts about these rings at a level allowing researchers and graduate students to gain entry to the field. This pays off by giving new insights into some of the outstanding open questions about quasi-Frobenius rings.

Our approach begins by extending many earlier results to a much wider class of rings than heretofore investigated. We call these rings *mininjective*. The remarkable thing is that our general methods yield basic information about these rings that has been overlooked in more focused studies. We present important facts about mininjective rings that were not known even for the (much smaller class of) self-injective rings studied classically. Moreover, the methods we have developed unify and simplify what is known in this area of research and so bring researchers and graduate students up to the research level.

The required background knowledge of noncommutative rings can be found in texts such as T. Y. Lam's *Lectures on Modules and Rings* (Springer-Verlag, 1998) or F. Anderson and K. R. Fuller's *Rings and Categories of Modules*

(Springer-Verlag, 1992). Appendices are included that develop the basic facts about Morita equivalence and about perfect, semiperfect, and semiregular rings to a level sufficient for our purposes. For more detailed information, the reader is referred to C. Faith's *Algebra II, Ring Theory* (Springer-Verlag, 1976), F. Kasch's *Modules and Rings* (Academic Press, 1982), and R. Wisbauer's *Foundations of Module and Ring Theory* (Gordon and Breach, 1991).

Consider the following four theorems: A ring R is quasi-Frobenius if it satisfies any of the following conditions:

1. R is right (or left) artinian and, if $\{e_1, e_2, \ldots, e_n\}$ is a basic set of primitive idempotents of R, there exists a permutation σ of $\{1, 2, \ldots, n\}$ such that $soc(Re_k) \cong Re_{\sigma k}/Je_{\sigma k}$ and $soc(e_{\sigma k}R) \cong e_k R/e_k J$.
2. R is right (or left) perfect and left and right self-injective.
3. Every right (or left) R-module embeds in a free module.
4. R is right (or left) noetherian and every one-sided ideal of R is an annihilator.

Here (1) and (4) are essentially due to Nakayama, (2) is due to Osofsky, and (3) is due to Faith and Walker. There are numerous other equivalent conditions that a ring is quasi-Frobenius; we have chosen these because of their relevance to three problems that we will refer to as follows:

The Faith conjecture: Every left (or right) perfect, right self-injective ring is quasi-Frobenius.

The FGF-conjecture: Every right FGF ring is quasi-Frobenius. (A ring is called a right *FGF ring* if every finitely generated right module embeds in a free module.)

The Faith–Menal conjecture: Every strongly right Johns ring is quasi-Frobenius. (A ring R is called *right Johns* if R is right noetherian and every right ideal is an annihilator; and R is called *strongly right Johns* if the matrix ring $M_n(R)$ is right Johns for all $n \geq 1$.)

This book reviews recent work on these conjectures and provides some new results. One of the main purposes of the monograph is to clearly outline both new and old methods for attacking these problems in an easily accessible format.

The chapter dependencies are pretty much in the order they appear, except that neither Chapter 3 nor Chapter 4 depends very much on the other. The required background about injectivity and continuity is developed in Chapter 1. A ring is called right *mininjective* if every isomorphism between simple right ideals is given by multiplication, and the basic properties of these rings are derived in Chapter 2. The profound consequences of insisting that a right mininjective ring is semiperfect are investigated in Chapter 3, leading to some important subclasses (the right minfull rings and the right min-PF rings) that are referred to throughout the book. Chapter 4 varies the theme of Chapters 2 and 3 and deals with the *min-CS rings* in which every simple right ideal is essential in a direct summand.

Two important subclasses of mininjective rings are introduced in Chapters 5 and 6. The right *principally injective* rings (for which every linear map from a principal right ideal to the ring is given by multiplication) are described in Chapter 5 and are shown to be closely related to the right FP-injective rings. This motivates the study of the *FP rings* [semiperfect, right FP-injective with essential right (or left) socle] as a generalization of the well-known class of pseudo-Frobenius rings. A ring is called right *simple injective* if every linear map with simple image from a right ideal to the ring is given by multiplication. These rings are investigated in Chapter 6 and are used to study *dual rings* (for which every one-sided ideal is an annihilator) and right *Ikeda–Nakayama* rings [for which $\mathtt{l}(A \cap B) = \mathtt{l}(A) + \mathtt{l}(B)$ for all right ideals A and B of R, where $\mathtt{l}(X)$ denotes the left annihilator].

A ring is called a right *C2 ring* if every right ideal that is isomorphic to a direct summand is itself a direct summand. In Chapter 7 the FGF-conjecture is shown to be closely related to these right C2 rings: A ring is quasi-Frobenius if every matrix ring over it is a C2 ring and every 2-generated right module embeds in a free module. This implies several important results in the literature and leads to a reformulation of the conjecture: The FGF-conjecture is true if and only if every right FGF ring is a right C2 ring. More recently, extensive work on the conjecture has been carried out by Gómez Pardo and Guil Asensio. They show that a right FGF ring is quasi-Frobenius if it is a right CS ring (every right ideal is essential in a direct summand). This in turn stems from their more general result: Every right Kasch, right CS ring has a finitely generated essential right socle, generalizing (and adapting the proof of) a well-known theorem of Osofsky in the right self-injective case.

The Faith–Menal conjecture is investigated in Chapter 8; in Chapter 9 a generic example is constructed to study the Faith conjecture and provide a source of examples of many of the rings studied in the book.

Of course a book like this rests on the research of many mathematicians, and it is a pleasure to acknowledge all these contributions. Special thanks go to Esperanza Sánchez Campos who gave the entire manuscript a thorough reading, made many useful suggestions, and caught a multitude of typographical errors. In addition, we thank Joanne Longworth for many consultations about the computer. We also acknowledge the support of the Ohio State University, the University of Calgary, NSERC Grant A-8075, and a Killam Resident Fellowship at the University of Calgary. Finally, we thank our families for their constant support during the time this book was being written.