# An Interactive Introduction to Mathematical Analysis

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# Chapter 1
# The Emergence of Rigorous Calculus

## 1.1 What Is Mathematical Analysis?

Mathematical analysis[8] is the critical and careful study of calculus with an emphasis on understanding of its basic principles. As opposed to *discrete* mathematics or *finite* mathematics, mathematical analysis can be thought of as being a form of *infinite* mathematics. As such, it must rank as one of the greatest, most powerful, and most profound creations of the human mind.

> The infinite! No other question has ever moved so profoundly the spirit of man — David Hilbert (1921).

Now, as you may expect, great, profound, and powerful thoughts do not often appear overnight. In fact, it took the best part of 2500 years from the time the first calculus-like problems tormented Pythagoras, until the first really solid foundations of mathematical analysis were laid in the nineteenth century. During the seventeenth and eighteenth centuries calculus blossomed, becoming an important branch of mathematics and, at the same time, a powerful tool, able to describe such physical phenomena as the motion of the planets, the stability of a spinning top, the behavior of a wave, and the laws of electrodynamics. This period saw the emergence of almost all of the concepts that one might expect to see in an elementary calculus course today.

But if the blossoms of calculus were formed during the seventeenth and eighteenth centuries, then its roots were formed during the nineteenth. Calculus underwent a revolution during the nineteenth century, a revolution in which its fundamental ideas were revealed and in which its underlying theory was properly understood for the first time. In this revolution, calculus was rewritten from its foundations by a small band of pioneers, among whom were Bernhard Bolzano, Augustin Cauchy, Karl Weierstrass, Richard Dedekind, and Georg Cantor. You will see their names repeatedly in this book, for it was largely as a result of their efforts that the subject that we know today as *mathematical analysis* was born. Their work enabled us to appreciate the nature of our number system and gave us our first solid understanding of the concepts of limit, continuity, derivative, and

[8] Note to instructors: This chapter is not designed for in-class teaching. It is intended to be a reading assignment, possibly in conjunction with other material that the student can find in the library.

integral. This is the great and profound theory to which you, the reader of this book, are heir.

In this chapter we shall focus on three earth-shaking events that have taken place during the past 2500 years and which helped to pave the way for the emergence of rigorous mathematics as we know it today. These events are sometimes known as the **Pythagorean crisis**, the **Zeno crisis**, and the **set theory crisis**.

## 1.2 The Pythagorean Crisis

In about 500 B.C.E. an individual in the Pythagorean school noticed that, according to the Greek concepts of number and length, it is impossible to compare the length of a side of a square with the length of its diagonal. The Greek concept of length required that, in order to compare two line segments $AB$ and $CD$, we need to be able to find a measuring rod that fits exactly a whole number of times into each of them. If, for example, the measuring rod fits $6$ times into $AB$ and $10$ times into $CD$, as shown in Figure 1.1, then we have

$$\frac{AB}{CD} = \frac{6}{10}.$$

More generally, if the measuring rod fits exactly $m$ times into $AB$ and exactly $n$

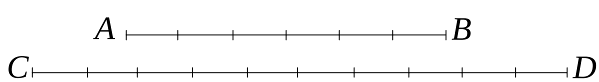


Figure 1.1

times into $CD$, then we have

$$\frac{AB}{CD} = \frac{m}{n}.$$

Note that this kind of comparison requires that the ratio of any two lengths must be a rational number.

The crisis came when the young Pythagorean drew a square with a side of one unit as shown in Figure 1.2 and applied the theorem of Pythagoras to find the length of the diagonal. As we know, the length of this diagonal is $\sqrt{2}$ units. From the fact that the number $\sqrt{2}$ is irrational he concluded that the equation

$$\frac{\sqrt{2}}{1} = \frac{m}{n}$$

is impossible if $m$ and $n$ are integers and that, consequently, it is impossible to compare the side of this square with its diagonal.

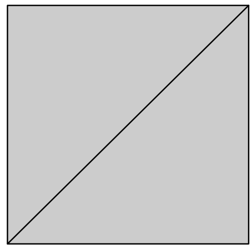


Figure 1.2

From our standpoint today, we can see that this discovery reveals the inadequacy of the rational number system and of the Greek concept of length; but to them, the discovery was a real shocker. Just how much of a shock it was can be gauged from the writings of the Greek philosopher Proclus, who tells us that the Pythagorean who made this terrible discovery suffered death by shipwreck as a punishment for it.

## 1.3 The Zeno Crisis

### 1.3.1 The Paradoxes of Zeno

In the fifth century B.C.E., Zeno of Elea came up with four innocent-sounding statements that plagued the philosophers all the way up to the time of Bolzano and Cauchy early in the nineteenth century. These four statements are known as the **paradoxes of Zeno**, and the first three of these appear in Bell [4] as follows:

1. *Motion is impossible, because whatever moves must reach the middle of its course before it reaches the end; but before it has reached the middle, it must have reached the quarter mark, and so on, indefinitely. Hence the motion can never start.*
2. *Achilles running to overtake a crawling tortoise ahead of him can never overtake it, because he must first reach the place from which the tortoise started; when Achilles reaches that place, the tortoise has departed and so is still ahead. Repeating the argument, we easily see that the tortoise will always be ahead.*
3. *A moving arrow at any instant is either at rest or not at rest, that is, moving. If the instant is indivisible, the arrow cannot move, for if it did, the instant would immediately be divided. But time is made up of instants. As the arrow cannot move in any one instant, it cannot move in any time. Hence it always remains at rest.*

Much has been said about these paradoxes, and, quite obviously, we are not going to do them justice here. But let's talk about the third paradox for a moment. At any one instant of time, the arrow does not move. Does that really mean that the arrow will not find its target? Would Zeno have been prepared to stand in front of the arrow? We think not. Then what was Zeno trying to tell us? Zeno's statement warns us that velocity can be meaningful in any physical sense only as an *average velocity over a period of time.* If an arrow covers a distance of 60 feet during the course of a second, we can say that the arrow has an average velocity of 60 feet per second. But Zeno's statement warns us that our senses can make nothing out of a notion of *velocity of the arrow at any one instant.*

### 1.3.2 Stating Zeno's Third Paradox in Terms of Slope

To state Zeno's third paradox in terms of slope, we shall suppose that $A$ is the point $(x_1, f(x_1))$ on the graph of a function $f$, and that $B$ is some other point $(x_1 + \Delta x, f(x_1 + \Delta x))$, as shown in Figure 1.3. As usual, the slope of the line

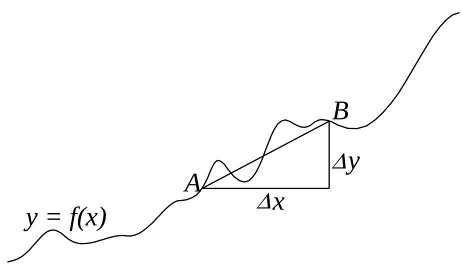


Figure 1.3

segment $AB$ is defined to be to be the ratio $\Delta y/\Delta x$, where

$$\Delta y = f(x_1 + \Delta x) - f(x_1).$$

This ratio $\Delta y/\Delta x$ is the average slope of the graph of $f$ between the points $A$ and $B$. However, Zeno's third paradox serves as a warning that there is no obvious physical meaning to the notion of *slope of the graph at the point* $A$.

"But" you may ask, "isn't this what calculus is all about? Are the paradoxes of Zeno trying to tell us to abandon the idea of a derivative?" They are not. But what we should learn from these paradoxes is that if we want to *define* the derivative of the function $f$ at the point $A$ to be

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

then that's just fine with Zeno. Only we can't blame Zeno if this derivative that we have *defined* doesn't measure how the function $f$ increases at $A$, because, as Zeno quite rightly tells us, the function $f$ can't change its value at any one point. We may therefore think of Zeno's paradoxes as telling us that (referring to Figure 1.3) even though we may speak of the slope $\frac{\Delta y}{\Delta x}$ of the line segment $AB$, and even though we may *define* the derivative of $f$ at $A$ and call it $\frac{dy}{dx}$ and have

$$\frac{\Delta y}{\Delta x} \to \frac{dy}{dx} \quad \text{as} \quad \Delta x \to 0,$$

we may not think of $\frac{dy}{dx}$ as the ratio of two quantities $dy$ and $dx$, the amounts by which $y$ and $x$ increase at the point $A$, because, as Zeno quite rightly tells us, there are *no* increases in $y$ and $x$ at the point $A$.

### 1.3.3 The Rigorous Reformulation

Mathematics prior to the dawn of the nineteenth century was much less precise than mathematics as we know it today. The core of pre-nineteenth century mathematics was the calculus that had been developed by Newton, Leibniz, and others during the seventeenth century. That calculus represented a magnificent contribution. It gave us the notation for derivatives and integrals that we still use today and provided a mathematical basis for the understanding of such physical phenomena as the motion of the planets, the motion of a spinning top and the vibration of a violin string. But the calculus of Newton and Leibniz did not rest on a solid foundation.

The problem with Newtonian calculus is that it was not based on an adequate theory of limits. In fact, prior to the nineteenth century, there was not much understanding that calculus needs to be based on a theory of limits at all. Nor was there much understanding of the nature of the number system $\mathbf{R}$ and the role of what we call today the *completeness* of the number system $\mathbf{R}$. In a sense, the calculus of Newton and Leibniz did not pay sufficient heed to the paradoxes of Zeno. Although Newton and Leibniz themselves may have had some appreciation of the fundamental ideas upon which the concepts of derivative and integral depend, many of those who followed them did not. Until the end of the eighteenth century the majority of mathematicians based their work upon an impossible mythology. During this time, proofs of theorems in calculus commonly depended on a notion of "infinitely small" numbers, numbers that were zero for some purposes yet not for others. These were known as *evanescent numbers, differentials,* or *infinitesimals,* and, undeniably, their use provided a beautiful, revealing, and elegant way of looking at many of the important theorems of calculus. Even today we like to use the notion of an infinitesimal to motivate some of the theorems in calculus, and scientists use them even more frequently than mathematicians. But it is one thing to use the idea of an infinitesimal to *motivate* a theory, and it is quite another matter to base virtually the entire theory upon them. Today, the concept of an infinitesimal can actually be made precise in a modern mathematical theory that is known as *nonstandard analysis*, but there was no precision in the way infinitesimals were used in the eighteenth century.

During the eighteenth century, the voices of critics began to be heard. In 1733, Voltaire [32] described calculus as

> *The art of numbering and measuring exactly a thing whose existence cannot be conceived.*

Then, in 1734, Bishop George Berkeley, the philosopher, wrote an essay, Berkeley [5], in which he rebuked the mathematicians for the weak foundations upon which their calculus had been based, and he no doubt took great pleasure in asking

> *Whether the object, principles, and inferences of the modern analysis are more distinctly conceived, or more evidently deduced than religious mysteries and points of faith.*

Some mathematicians composed weak answers to Berkeley's criticism, and others tried vainly to make sense of the idea of infinitely small numbers, but it was not until the early nineteenth century that any real progress was made. The turning point came with the work of Bernhard Bolzano, who gave us the first coherent definition of limits and continuity and the first understanding of the need for a complete number system. Then came the work of Cauchy, Weierstrass, Dedekind, and Cantor that placed calculus on a rigorous foundation and settled many important questions about the nature of the number system $\mathbf{R}$. The work of these pioneers has made possible the understanding that we have promised you.

## 1.4 The Set Theory Crisis

Following the work of the nineteenth-century pioneers, the mathematical community began to believe that true understanding was at last within its grasp. All of the fundamental concepts seemed to be rooted solidly in Cantor's theory of sets. But the collective sigh of relief had hardly died away when a new kind of paradox burst upon the scene. In 1897, the Italian mathematician Burali-Forti discovered what is known today as the **Burali-Forti paradox**, which shows that there are serious flaws in Cantor's theory of sets, upon which our understanding of the real number system had been based. Then, a few years later, Bertrand Russell discovered his famous paradox. Like Burali-Forti's paradox, Russell's paradox demonstrates the presence of flaws in Cantor's set theory.

To see just how much these paradoxes stunned the mathematical community, one might want to look at Frege [9], *Grundgesetze der Arithmetik* (*The Fundamental Laws of Arithmetic*), which was written by the German philosopher Gottlob Frege and published in two volumes, the first in 1893 and the second in 1903. This book was Frege's life work, and it was his pride and joy. He had bestowed upon the mathematical community the first sound analysis of the meaning of number and the laws of arithmetic and, although the book is quite technical in places, it is worth skimming through, if only to see the sarcastic way in which Frege speaks of the "stupidity" of those who had come before him. An example of this sarcasm is Frege's description of his attempt to induce other mathematicians to tell him what the number *one* means. "One object," would be the reply. "Very well," answered Frege, "I choose the *moon*! Now I ask you please to tell me: *Is one plus one still equal to two*?" As things turned out, the second volume of Frege's book came out just after Russell had sent Frege his famous paradox. There was just enough space at the end of Frege's book for the following acknowledgment:

> *A scientist can hardly encounter anything more undesirable than to have the foundation collapse just as the work is finished. I was put in this position by a letter from Mr. Bertrand Russell when the work was almost through the press.*

As Frege said, the foundation collapsed. It would not be stretching the truth too much to say that all of mathematics perished in the fire storm that was ignited by the paradoxes of Russell and Burali-Forti. The mathematics that we know today is what emerged from that storm like a phoenix from the ashes, and it depends upon a new theory of sets that is known as **Zermelo-Fraenkel set theory** which was developed in the first few decades of the twentieth century. Within the framework of Zermelo-Fraenkel set theory, we can once again make use of Frege's important work.

One question that remains is whether we are now safe from new paradoxes that might ignite a new fire storm, and the answer is that we don't know. A theorem of Gödel guarantees that, unless someone actually discovers a new paradox that destroys Zermelo-Fraenkel set theory, we shall never know whether such a paradox exists. Thus it is entirely possible that you, the reader of this book, may stumble upon a snag that shows that mathematics as we know it does not work. But don't hold your breath. The chances of your encountering a new paradox are very remote.