New Directions in Hopf Algebras
MSRI Publications
Volume **43**, 2002

Pointed Hopf Algebras

NICOLÁS ANDRUSKIEWITSCH AND HANS-JÜRGEN SCHNEIDER

Abstract. This is a survey on pointed Hopf algebras over algebraically closed fields of characteristic 0. We propose to classify pointed Hopf algebras A by first determining the graded Hopf algebra $\operatorname{gr} A$ associated to the coradical filtration of A. The A_0-coinvariants elements form a braided Hopf algebra R in the category of Yetter–Drinfeld modules over the coradical $A_0 = \Bbbk\Gamma$, Γ the group of group-like elements of A, and $\operatorname{gr} A \simeq R\#A_0$. We call the braiding of the primitive elements of R the infinitesimal braiding of A. If this braiding is of Cartan type [AS2], then it is often possible to determine R, to show that R is generated as an algebra by its primitive elements and finally to compute all deformations or liftings, that is pointed Hopf algebras such that $\operatorname{gr} A \simeq R\#\Bbbk\Gamma$. In the last chapter, as a concrete illustration of the method, we describe explicitly all finite-dimensional pointed Hopf algebras A with abelian group of group-likes $G(A)$ and infinitesimal braiding of type A_n (up to some exceptional cases). In other words, we compute all the liftings of type A_n; this result is our main new contribution in this paper.

Contents

2000 *Mathematics Subject Classification.* Primary: 17B37; Secondary: 16W30.

Key words and phrases. Pointed Hopf algebras, finite quantum groups.

This work was partially supported by ANPCyT, Agencia Córdoba Ciencia, CONICET, DAAD, the Graduiertenkolleg of the Math. Institut (Universität München) and Secyt (UNC).

Introduction

A Hopf algebra A over a field $\Bbbk$ is called *pointed* [Sw], [M1], if all its simple left or right comodules are one-dimensional. The coradical A_0 of A is the sum of all its simple subcoalgebras. Thus A is pointed if and only if A_0 is a group algebra.

We will always assume that the field $\Bbbk$ is algebraically closed of characteristic 0 (although several results of the paper hold over arbitrary fields).

It is easy to see that A is pointed if it is generated as an algebra by group-like and skew-primitive elements. In particular, group algebras, universal enveloping algebras of Lie algebras and the q-deformations of the universal enveloping algebras of semisimple Lie algebras are all pointed.

An essential tool in the study of pointed Hopf algebras is the *coradical filtration*

$$A_0 \subset A_1 \subset \cdots \subset A, \qquad \bigcup_{n\geq 0} A_n = A$$

of A. It is dual to the filtration of an algebra by the powers of the Jacobson radical. For pointed Hopf algebras it is a Hopf algebra filtration, and the associated graded Hopf algebra $\operatorname{gr} A$ has a Hopf algebra projection onto $A_0 = \Bbbk\Gamma, \Gamma = G(A)$ the group of all group-like elements of A. By a theorem of Radford [Ra], $\operatorname{gr} A$ is a biproduct

$$\operatorname{gr} A \cong R\#\Bbbk\Gamma,$$

where R is a graded braided Hopf algebra in the category of left Yetter–Drinfeld modules over $\Bbbk\Gamma$ [AS2].

This decomposition is an analog of the theorem of Cartier–Kostant–Milnor–Moore on the semidirect product decomposition of a cocommutative Hopf algebra into an infinitesimal and a group algebra part.

The vector space $V = P(R)$ of the primitive elements of R is a Yetter–Drinfeld submodule. We call its braiding

$$c : V \otimes V \to V \otimes V$$

the **infinitesimal braiding** of A. The infinitesimal braiding is the key to the structure of pointed Hopf algebras.

The subalgebra $\mathfrak{B}(V)$ of R generated by V is a braided Hopf subalgebra. As an algebra and coalgebra, $\mathfrak{B}(V)$ only depends on the infinitesimal braiding of V. In his thesis [N] published in 1978, Nichols studied Hopf algebras of the form $\mathfrak{B}(V)\#\Bbbk\Gamma$ under the name of bialgebras of type one. We call $\mathfrak{B}(V)$ the *Nichols algebra* of V. These Hopf algebras were found independently later by Woronowicz [Wo] and other authors.

Important examples of Nichols algebras come from quantum groups [Dr1]. If $\mathfrak{g}$ is a semisimple Lie algebra, $U_q^{\geq 0}(\mathfrak{g})$, q not a root of unity, and the finite-dimensional Frobenius–Lusztig kernels $\mathfrak{u}_q^{\geq 0}(\mathfrak{g})$, q a root of unity of order N, are both of the form $\mathfrak{B}(V)\#\Bbbk\Gamma$ with $\Gamma = \mathbb{Z}^\theta$ resp. $(\mathbb{Z}/(N))^\theta, \theta \geq 1$. ([L3], [Ro1], [Sbg], and [L2], [Ro1], [Mu]) (assuming some technical conditions on N).

In general, the classification problem of pointed Hopf algebras has three parts:

(1) Structure of the Nichols algebras $\mathfrak{B}(V)$.
(2) The lifting problem: Determine the structure of all pointed Hopf algebras A with $G(A) = \Gamma$ such that $\operatorname{gr} A \cong \mathfrak{B}(V)\#\Bbbk\Gamma$.
(3) Generation in degree one: Decide which Hopf algebras A are generated by group-like and skew-primitive elements, that is $\operatorname{gr} A$ is generated in degree one.

We conjecture that all finite-dimensional pointed Hopf algebras over an algebraically closed field of characteristic 0 are indeed generated by group-like and skew-primitive elements.

In this paper, we describe the steps of this program in detail and explain the positive results obtained so far in this direction. It is not our intention to give a complete survey on all aspects of pointed Hopf algebras.

We will mainly report on recent progress in the classification of pointed Hopf algebras with *abelian* group of group-like elements.

If the group Γ is abelian, and V is a finite-dimensional Yetter–Drinfeld module, then the braiding is given by a family of non-zero scalars $q_{ij} \in \Bbbk$, $1 \leq i \leq \theta$, in the form

$$c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i, \text{ where } x_1, \dots, x_\theta \text{ is a basis of } V.$$

Moreover there are elements $g_1, \dots, g_\theta \in \Gamma$, and characters $\chi_1, \dots, \chi_\theta \in \widehat{\Gamma}$ such that $q_{ij} = \chi_j(g_i)$. The group acts on x_i via the character χ_i, and x_i is a g_i-homogeneous element with respect to the coaction of Γ. We introduced braidings of Cartan type [AS2] where

$$q_{ij}q_{ji} = q_{ii}^{a_{ij}}, 1 \leq i, j \leq \theta, \text{ and } (a_{ij}) \text{ is a generalized Cartan matrix.}$$

If (a_{ij}) is a Cartan matrix of finite type, then the algebras $\mathfrak{B}(V)$ can be understood as twisting of the Frobenius–Lusztig kernels $\mathfrak{u}^{\geq 0}(\mathfrak{g})$, $\mathfrak{g}$ a semisimple Lie algebra.

By deforming the quantum Serre relations for simple roots which lie in two different connected components of the Dynkin diagram, we define finite-dimensional pointed Hopf algebras $\mathfrak{u}(\mathcal{D})$ in terms of a "linking datum $\mathcal{D}$ of finite Cartan type" [AS4]. They generalize the Frobenius–Lusztig kernels $\mathfrak{u}(\mathfrak{g})$ and are liftings of $\mathfrak{B}(V)\#\Bbbk\Gamma$.

In some cases linking data of finite Cartan type are general enough to obtain complete classification results.

For example, if $\Gamma = (\mathbb{Z}/(p))^s$, p a prime > 17 and $s \geq 1$, we have determined the structure of all finite-dimensional pointed Hopf algebras A with $G(A) \simeq \Gamma$. They are all of the form $\mathfrak{u}(\mathcal{D})$ [AS4].

Similar data allow a classification of infinite-dimensional pointed Hopf algebras A with abelian group $G(A)$, without zero divisors, with finite Gelfand–Kirillov dimension and semisimple action of $G(A)$ on A, in the case when the infinitesimal braiding is "positive" [AS5].

But the general case is more involved. We also have to deform the root vector relations of the $\mathfrak{u}(\mathfrak{g})'s$.

The structure of pointed Hopf algebras A with *non-abelian* group $G(A)$ is largely unknown. One basic open problem is to decide which finite groups appear as groups of group-like elements of finite-dimensional pointed Hopf algebras which are link-indecomposable in the sense of [M2]. In our formulation, this problem is the main part of the following question: given a finite group Γ, determine all Yetter–Drinfeld modules V over $\Bbbk\Gamma$ such that $\mathfrak{B}(V)$ is finite-dimensional. On the one hand, there are a number of severe constraints on V [Gñ3]. See also the exposition in [A, 5.3.10]. On the other hand, it is very hard to prove the finiteness of the dimension, and in fact this has been done only for a few examples [MiS], [FK], [FP] which are again related to root systems. The examples over the symmetric groups in [FK] were introduced to describe the cohomology ring of the flag variety. At this stage, the main difficulty is to decide when certain Nichols algebras over non-abelian groups, for example the symmetric groups $\mathbb{S}_n$, are finite-dimensional.

The last chapter provides a concrete illustration of the theory explained in this paper. We describe explicitly all finite-dimensional pointed Hopf algebras with abelian group $G(A)$ and infinitesimal braiding of type A_n (up to some exceptional cases). The main results in this chapter are new, and complete proofs are given. The only cases which were known before are the easy case A_1 [AS1], and A_2 [AS3].

The new relations concern the root vectors $e_{i,j}$, $1 \le i < j \le n+1$. The relations $e_{i,j}^N = 0$ in $\mathfrak{u}_q^{\geq 0}(sl_{n+1})$, q a root of unity of order N, are replaced by

$$e_{i,j}^N = u_{i,j} \text{ for a family } u_{i,j} \in \Bbbk\Gamma, 1 \le i < j \le n+1,$$

depending on a family of free parameters in $\Bbbk$. See Theorem 6.25 for details.

Lifting of type B_2 was treated in [BDR].

To study the relations between a filtered object and its associated graded object is a basic technique in modern algebra. We would like to stress that finite-dimensional pointed Hopf algebras enjoy a remarkable rigidity; it is seldom the case that one is able to describe precisely all the liftings of a graded object, as in this context.

Acknowledgements. We would like to thank Jacques Alev, Matías Graña and Eric Müller for suggestions and stimulating discussions.

Conventions. As said above, our ground field $\Bbbk$ is algebraically closed field of characteristic 0. Throughout, "Hopf algebra" means "Hopf algebra with bijective antipode". Δ, $\mathcal{S}$, ε, denote respectively the comultiplication, the antipode, the counit of a Hopf algebra.

We denote by $\tau : V \otimes W \to W \otimes V$ the usual transposition, that is $\tau(v \otimes w) = w \otimes v$.

We use Sweedler's notation for the comultiplication and coaction; but, to avoid confusions, we use the following variant for the comultiplication of a braided Hopf algebra R: $\Delta_R(r) = r^{(1)} \otimes r^{(2)}$.

1. Braided Hopf Algebras

1.1. Braided categories. Braided Hopf algebras play a central rôle in this paper. Although we have tried to minimize the use of categorical language, we briefly and informally recall the notion of a braided category which is the appropriate setting for braided Hopf algebras.

Braided categories were introduced in [JS]. We refer to [Ka, Ch. XI, Ch. XIII] for a detailed exposition. There is a hierarchy of categories with a tensor product functor:

(a) A *monoidal* or *tensor* category is a collection $(\mathcal{C}, \otimes, a, \mathbb{I}, l, r)$, where

- $\mathcal{C}$ is a category and $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a functor,
- $\mathbb{I}$ is an object of $\mathcal{C}$, and
- $a_{V,W,U} : V \otimes (W \otimes U) \to (V \otimes W) \otimes U$, $l_V : V \to V \otimes \mathbb{I}$, $r_V : V \to \mathbb{I} \otimes V$, V, W, U objects in $\mathcal{C}$, are natural isomorphisms;

such that the so-called "pentagon" and "triangle" axioms are satisfied, see [Ka, Ch. XI, (2.6) and (2.9)]. These axioms essentially express that the tensor product of a finite number of objects is well-defined, regardless of the place where parentheses are inserted; and that $\mathbb{I}$ is a unit for the tensor product.

(b) A *braided (tensor)* category is a collection $(\mathcal{C}, \otimes, a, \mathbb{I}, l, r, c)$, where

- $(\mathcal{C}, \otimes, a, \mathbb{I}, l, r)$ is a monoidal category and
- $c_{V,W} : V \otimes W \to W \otimes V$, V, W objects in $\mathcal{C}$, is a natural isomorphism;

such that the so-called "hexagon" axioms are satisfied, see [Ka, Ch. XIII, (1.3) and (1.4)]. A very important consequence of the axioms of a braided category is the following equality for any objects V, W, U:

$$(c_{V,W} \otimes \mathrm{id}\,_U)(\mathrm{id}\,_V \otimes c_{U,W})(c_{U,V} \otimes \mathrm{id}\,_W) = (\mathrm{id}\,_W \otimes c_{U,V})(c_{U,W} \otimes \mathrm{id}\,_V)(\mathrm{id}\,_U \otimes c_{V,W}), \tag{1–1}$$

see [Ka, Ch. XIII, (1.8)]. For simplicity we have omitted the associativity morphisms.

(c) A *symmetric* category is a braided category where $c_{V,W} c_{W,V} = \mathrm{id}\,_{W \otimes V}$ for all objects V, W. Symmetric categories have been studied since the pioneering work of Mac Lane.

(d) A left dual of an object V of a monoidal category, is a triple $(V^*, \mathrm{ev}_V, b_V)$, where V^* is another object and $\mathrm{ev}_V : V^* \otimes V \to \mathbb{I}$, $b_V : \mathbb{I} \to V \otimes V^*$ are morphisms such that the compositions

$$V \longrightarrow \mathbb{I} \otimes V \xrightarrow{b_V \otimes \mathrm{id}_V} V \otimes V^* \otimes V \xrightarrow{\mathrm{id}_V \otimes \mathrm{ev}_V} V \otimes \mathbb{I} \longrightarrow V$$

and

$$V^* \longrightarrow V^* \otimes \mathbb{I} \xrightarrow{\mathrm{id}_{V^*} \otimes b_V} V^* \otimes V \otimes V^* \xrightarrow{\mathrm{ev}_V \otimes \mathrm{id}_{V^*}} \mathbb{I} \otimes V^* \longrightarrow V^*$$

are, respectively, the identity of V and V^*. A braided category is *rigid* if any object V admits a left dual [Ka, Ch. XIV, Def. 2.1].

1.2. Braided vector spaces and Yetter–Drinfeld modules. We begin with the fundamental

Definition 1.1. Let V be a vector space and $c : V \otimes V \to V \otimes V$ a linear isomorphism. Then (V, c) is called a *braided vector space*, if c is a solution of the *braid equation*, that is

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c). \qquad (1\text{–}2)$$

It is well-known that the braid equation is equivalent to the *quantum Yang–Baxter equation*:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \qquad (1\text{–}3)$$

Here we use the standard notation: $R_{13} : V \otimes V \otimes V \to V \otimes V \otimes V$ is the map given by $\sum_j r_j \otimes \mathrm{id} \otimes r^j$, if $R = \sum_j r_j \otimes r^j$. Similarly for R_{12}, R_{23}.

The equivalence between solutions of (1–2) and solutions of (1–3) is given by the equality $c = \tau \circ R$. For this reason, some authors call (1–2) the quantum Yang–Baxter equation.

An easy and for this paper important example is given by a family of non-zero scalars $q_{ij} \in \Bbbk, i, j \in I$, where V is a vector space with basis $x_i, i \in I$. Then

$$c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i, \text{for all } i, j \in I$$

is a solution of the braid equation.

Examples of braided vector spaces come from braided categories. In this article, we are mainly concerned with examples related to the notion of Yetter–Drinfeld modules.

Definition 1.2. Let H be a Hopf algebra. A (left) *Yetter–Drinfeld module* V over H is simultaneously a left H-module and a left H-comodule satisfying the compatibility condition

$$\delta(h.v) = h_{(1)} v_{(-1)} \mathcal{S} h_{(3)} \otimes h_{(2)}.v_{(0)}, \qquad v \in V,\ h \in H. \qquad (1\text{–}4)$$

We denote by ${}^H_H\mathcal{YD}$ the category of Yetter–Drinfeld modules over H; the morphisms in this category preserve both the action and the coaction of H. The category ${}^H_H\mathcal{YD}$ is a braided monoidal category; indeed the tensor product of two Yetter–Drinfeld modules is again a Yetter–Drinfeld module, with the usual tensor product module and comodule structure. The compatibility condition (1–4) is not difficult to verify.

For any two Yetter–Drinfeld-modules M and N, the braiding $c_{M,N} : M \otimes N \to N \otimes M$ is given by

$$c_{M,N}(m \otimes n) = m_{(-1)}.n \otimes m_{(0)}, \qquad m \in M,\, n \in N. \tag{1–5}$$

The subcategory of ${}^H_H\mathcal{YD}$ consisting of finite-dimensional Yetter–Drinfeld modules is rigid. Namely, if $V \in {}^H_H\mathcal{YD}$ is finite-dimensional, the dual $V^* = \operatorname{Hom}(V, \Bbbk)$ is in ${}^H_H\mathcal{YD}$ with the following action and coaction:

- $(h \cdot f)(v) = f(\mathcal{S}(h)v)$ for all $h \in H$, $f \in V^*$, $v \in V$.
- If $f \in V^*$, then $\delta(f) = f_{(-1)} \otimes f_{(0)}$ is determined by the equation

$$f_{(-1)} f_{(0)}(v) = \mathcal{S}^{-1}(v_{-1}) f(v_0), \qquad v \in V.$$

Then the usual evaluation and coevaluation maps are morphisms in ${}^H_H\mathcal{YD}$.

Let V, W be two finite-dimensional Yetter–Drinfeld modules over H. We shall consider the isomorphism $\Phi : W^* \otimes V^* \to (V \otimes W)^*$ given by

$$\Phi(\varphi \otimes \psi)(v \otimes w) = \psi(v)\varphi(w), \qquad \varphi \in W^*, \psi \in V^*, v \in V, w \in W. \tag{1–6}$$

Remark 1.3. We see that a Yetter–Drinfeld module is a braided vector space. Conversely, a braided vector space (V, c) can be realized as a Yetter–Drinfeld module over some Hopf algebra H if and only if c is *rigid* [Tk1]. If this is the case, it can be realized in many different ways.

We recall that a Hopf bimodule over a Hopf algebra H is simultaneously a bimodule and a bicomodule satisfying all possible compatibility conditions. The category ${}^H_H\mathcal{M}^H_H$ of all Hopf bimodules over H is a braided category. The category ${}^H_H\mathcal{YD}$ is equivalent, as a braided category, to the category of Hopf bimodules. This was essentially first observed in [Wo] and then independently in [AnDe, Appendix], [Sbg], [Ro1].

If H is a finite-dimensional Hopf algebra, then the category ${}^H_H\mathcal{YD}$ is equivalent to the category of modules over the double of H [Mj1]. The braiding in ${}^H_H\mathcal{YD}$ corresponds to the braiding given by the "canonical" R-matrix of the double. In particular, if H is a semisimple Hopf algebra then ${}^H_H\mathcal{YD}$ is a semisimple category. Indeed, it is known that the double of a semisimple Hopf algebra is again semisimple.

The case of Yetter–Drinfeld modules over group algebras is especially important for the applications to pointed Hopf algebras. If $H = \Bbbk\Gamma$, where Γ is a group, then an H-comodule V is just a Γ-graded vector space: $V = \bigoplus_{g \in \Gamma} V_g$, where $V_g = \{v \in V \mid \delta(v) = g \otimes v\}$. We will write ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ for the category of Yetter–Drinfeld modules over $\Bbbk\Gamma$, and say that $V \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$ is a Yetter–Drinfeld module over Γ (when the field is fixed).

Remark 1.4. Let Γ be a group, V a left $\Bbbk\Gamma$-module, and a left $\Bbbk\Gamma$-comodule with grading $V = \bigoplus_{g\in\Gamma} V_g$. We define a linear isomorphism $c : V \otimes V \to V \otimes V$ by

$$c(x \otimes y) = gy \otimes x, \text{ for all } x \in V_g,\ g \in \Gamma,\ y \in V. \tag{1–7}$$

Then

(a) $V \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$ if and only if $gV_h \subset V_{ghg^{-1}}$ for all $g, h \in \Gamma$.
(b) If $V \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$, then (V, c) is a braided vector space.
(c) Conversely, if V is a faithful Γ-module (that is, if for all $g \in \Gamma, gv = v$ for all $v \in V$, implies $g = 1$), and if (V, c) is a braided vector space, then $V \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$.

Proof. (a) is clear from the definition.

By applying both sides of the braid equation to elements of the form $x \otimes y \otimes z, x \in V_g, y \in V_h, z \in V$, it is easy to see that (V, c) is a braided vector space if and only if

$$c(gy \otimes gz) = ghz \otimes gy, \text{ for all } g, h \in \Gamma,\ y \in V_h,\ z \in V. \tag{1–8}$$

Let us write $gy = \sum_{a\in\Gamma} x_a$, where $x_a \in V_a$ for all $a \in \Gamma$. Then $c(gy \otimes gz) = \sum_{a\in\Gamma} agz \otimes x_a$. Hence (1–8) means that $agz = ghz$, for all $z \in V$ and $a \in \Gamma$ such that the homogeneous component x_a is not zero. This proves (b) and (c). □

Remark 1.5. If Γ is abelian, a Yetter–Drinfeld module over $H = \Bbbk\Gamma$ is nothing but a Γ-graded Γ-module.

Assume that Γ is abelian and furthermore that the action of Γ is diagonalizable (this is always the case if Γ is finite). That is, $V = \bigoplus_{\chi\in\widehat{\Gamma}} V^{\chi}$, where $V^{\chi} = \{v \in V \mid gv = \chi(g)v \text{ for all } g \in \Gamma\}$. Then

$$V = \bigoplus_{g\in\Gamma,\chi\in\widehat{\Gamma}} V_g^{\chi}, \tag{1–9}$$

where $V_g^{\chi} = V^{\chi} \cap V_g$. Conversely, any vector space with a decomposition (1–9) is a Yetter–Drinfeld module over Γ. The braiding is given by

$$c(x \otimes y) = \chi(g)y \otimes x, \text{ for all } x \in V_g,\ g \in \Gamma,\ y \in V^{\chi},\ \chi \in \widehat{\Gamma}.$$

It is useful to characterize abstractly those braided vector spaces which come from Yetter–Drinfeld modules over groups or abelian groups. The first part of the following definition is due to M. Takeuchi.

Definition 1.6. Let (V, c) be a finite-dimensional braided vector space.

- (V,c) is of *group type* if there exist a basis $x_1,\dots,x_\theta$ of V and elements $g_i(x_j) \in V$ for all i,j such that

$$c(x_i \otimes x_j) = g_i(x_j) \otimes x_i, \qquad 1 \le i,j \le \theta; \tag{1–10}$$

  necessarily $g_i \in \mathrm{GL}(V)$.
- (V,c) is of *finite group type* (resp. of *abelian group type*) if it is of group type and the subgroup of $\mathrm{GL}(V)$ generated by $g_1,\dots,g_\theta$ is finite (resp. abelian).
- (V,c) is of *diagonal type* if V has a basis $x_1,\dots,x_\theta$ such that

$$c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i, \qquad 1 \le i,j \le \theta, \tag{1–11}$$

  for some q_{ij} in $\Bbbk$. The matrix (q_{ij}) is called the *matrix* of the braiding.
- If (V,c) is of diagonal type, then we say that it is *indecomposable* if for all $i \neq j$, there exists a sequence $i = i_1, i_2, \dots, i_t = j$ of elements of $\{1,\dots,\theta\}$ such that $q_{i_s,i_{s+1}} q_{i_{s+1},i_s} \neq 1$, $1 \le s \le t-1$. Otherwise, we say that the matrix is decomposable. We can also refer then to the components of the matrix.

If $V \in {}^\Gamma_\Gamma\mathcal{YD}$ is finite-dimensional with braiding c, then (V,c) is of group type by (1–5). Conversely, assume that (V,c) is a finite-dimensional braided vector space of group type. Let Γ be the subgroup of $\mathrm{GL}(V)$ generated by $g_1,\dots,g_\theta$. Define a coaction by $\delta(x_i) = g_i \otimes x_i$ for all i. Then V is a Yetter–Drinfeld module over Γ with braiding c by Remark 1.4 (c).

A braided vector space of diagonal type is clearly of abelian group type; it is of finite group type if the q_{ij}'s are roots of one.

1.3. Braided Hopf algebras. The notion of "braided Hopf algebra" is one of the basic features of braided categories. We will deal only with braided Hopf algebras in categories of Yetter–Drinfeld modules, mainly over a group algebra.

Let H be a Hopf algebra. First, the tensor product in ${}^H_H\mathcal{YD}$ allows us to define algebras and coalgebras in ${}^H_H\mathcal{YD}$. Namely, an algebra in the category ${}^H_H\mathcal{YD}$ is an associative algebra (R,m), where $m : R \otimes R \to R$ is the product, with unit $u : \Bbbk \to R$, such that R is a Yetter–Drinfeld module over H and both m and u are morphisms in ${}^H_H\mathcal{YD}$.

Similarly, a coalgebra in the category ${}^H_H\mathcal{YD}$ is a coassociative coalgebra (R,Δ), where $\Delta : R \to R \otimes R$ is the coproduct, with counit $\varepsilon : R \to \Bbbk$, such that R is a Yetter–Drinfeld module over H and both Δ and ε are morphisms in ${}^H_H\mathcal{YD}$.

Let now R, S be two algebras in ${}^H_H\mathcal{YD}$. Then the braiding $c : S \otimes R \to R \otimes S$ allows us to provide the Yetter–Drinfeld module $R \otimes S$ with a "twisted" algebra structure in ${}^H_H\mathcal{YD}$. Namely, the product in $R \otimes S$ is $m_{R\otimes S}$, defined as $(m_R \otimes m_S)(\mathrm{id} \otimes c \otimes \mathrm{id})$:

$$\begin{array}{ccc} R\otimes S\otimes R\otimes S & \longrightarrow & R\otimes S \\ {\scriptstyle \mathrm{id}\otimes c\otimes \mathrm{id}}\big\downarrow & & \big\uparrow{\scriptstyle =} \\ R\otimes R\otimes S\otimes S & \xrightarrow{m_R\otimes m_S} & R\otimes S. \end{array}$$

We shall denote this algebra by $R\underline{\otimes}S$. The difference with the usual tensor product algebra is the presence of the braiding c instead of the usual transposition τ.

DEFINITION 1.7. A *braided bialgebra in* ${}^H_H\mathcal{YD}$ is a collection $(R, m, u, \Delta, \varepsilon)$, where

- (R, m, u) is an algebra in ${}^H_H\mathcal{YD}$.
- (R, Δ, ε) is a coalgebra in ${}^H_H\mathcal{YD}$.
- $\Delta : R \to R\underline{\otimes}R$ and $\varepsilon : R \to \Bbbk$ are morphisms of algebras.

We say that it is a *braided Hopf algebra in* ${}^H_H\mathcal{YD}$ if in addition:

- The identity is convolution invertible in End (R); its inverse is the antipode of R.

A *graded* braided Hopf algebra in ${}^H_H\mathcal{YD}$ is a braided Hopf algebra R in ${}^H_H\mathcal{YD}$ provided with a grading $R = \bigoplus_{n\geq 0} R(n)$ of Yetter–Drinfeld modules, such that R is a graded algebra and a graded coalgebra.

REMARK 1.8. There is a non-categorical version of braided Hopf algebras, see [Tk1]. Any braided Hopf algebra in ${}^H_H\mathcal{YD}$ gives rise to a braided Hopf algebra in the sense of [Tk1] by forgetting the action and coaction, and preserving the multiplication, comultiplication and braiding. For the converse see [Tk1, Th. 5.7]. Analogously, one can define graded braided Hopf algebras in the spirit of [Tk1].

Let R be a finite-dimensional Hopf algebra in ${}^H_H\mathcal{YD}$. The dual $S = R^*$ is a braided Hopf algebra in ${}^H_H\mathcal{YD}$ with multiplication $\Delta^*_R\Phi$ and comultiplication $\Phi^{-1}m^*_R$, *cf.* (1–6); this is R^{*bop} in the notation of [AG, Section 2].

In the same way, if $R = \bigoplus_{n\geq 0} R(n)$ is a graded braided Hopf algebra in ${}^H_H\mathcal{YD}$ with finite-dimensional homogeneous components, then the graded dual $S = R^* = \bigoplus_{n\geq 0} R(n)^*$ is a graded braided Hopf algebra in ${}^H_H\mathcal{YD}$.

1.4. Examples. The quantum binomial formula. We shall provide many examples of braided Hopf algebras in Chapter 2. Here we discuss a very simple class of braided Hopf algebras.

We first recall the well-known quantum binomial formula. Let U and V be elements of an associative algebra over $\Bbbk[q]$, q an indeterminate, such that $VU = qUV$. Then

$$(U+V)^n = \sum_{1\leq i\leq n} \binom{n}{i}_q U^i V^{n-i}, \qquad \text{if } n \geq 1. \tag{1–12}$$

Here

$$\binom{n}{i}_q = \frac{(n)_q!}{(i)_q!(n-i)_q!}, \quad \text{where } (n)_q! = \prod_{1\leq i\leq n} (i)_q, \quad \text{and } (i)_q = \sum_{0\leq j\leq i-1} q^j.$$