Cambridge University Press & Assessment
978-0-521-81510-9 — Financial Derivatives
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Excerpt
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Introduction

This book is about risk and derivative securities. In our opinion, no one has described the issue more eloquently than Jorge Luis Borges, an intrepid Argentinian writer. He tells a fictional story of a lottery in ancient Babylonia. The lottery is peculiar because it is compulsory. All subjects are required to play and to accept the outcome. If they lose, they stand to lose their wealth, their lives, or their loved ones. If they win, they will get mountains of gold, the spouse of their choice, and other wonderful goodies.

It is easy to see how this story is a metaphor of our lives. We are shaped daily by doses of randomness. This is where the providential financial engineer intervenes. The engineer's thoughts are along the following lines: to confront all this randomness, one needs artificial randomness of opposite sign, called derivative securities. And the engineer calls the ratio of these two random quantities a hedge ratio.

Financial engineering is about combining the Tinker Toys of capital markets and financial institutions to create custom risk-return profiles for economic agents. An important element of the financial engineering process is the valuation of the Tinker Toys; this is the central ingredient this book provides.

We have written this book with a view to the following two objectives:

- to introduce readers with a modicum of mathematical background to the valuation of derivatives

- to give them the tools and intuition to expand upon these results when necessary

By and large, textbooks on derivatives fall into two categories: the first is targeted toward MBA students and advanced undergraduates, and the second aims at finance or mathematics PhD students. The former tend to score high on breadth of coverage but do not go in depth into any specific area of derivatives. The latter tend to be highly rigorous and therefore limit the audience. While this book is closer to the second category, it strives to simplify the mathematical presentation and make it accessible to a wider audience. Concepts such as measure, functional spaces, and Lebesgue integrals are avoided altogether in the interest of all those who have a good knowledge of mathematics but yet have not ventured into advanced mathematics.

The target audience includes advanced undergraduates in mathematics, economics, and finance; graduate students in quantitative finance master's programs as well as PhD students in the aforementioned disciplines; and practitioners afflicted with an interest in derivatives pricing and mathematical curiosity.

The book assumes elementary knowledge of finance at the level of the Brealey and Myers corporate finance textbook. Notions such as discounting, net present value, spot and forward rates, and basic option pricing in a binomial model should be familiar to the reader. However, very little knowledge of economics is assumed, as we develop the required utility theory from first principles.

The level of mathematical preparation required to get through this book successfully comprises knowledge of differential and integral calculus, probability, and statistics. In calculus, readers need to know basic differentiation and integration rules and Taylor series expansions, and should have some familiarity with differential equations. Readers should have had the standard year-long sequence in probability and statistics. This includes conventional, discrete, and continuous probability distributions and related notions, such as their moment generating functions and characteristic functions.

The outline runs as follows:

1. Chapter 1 provides readers with the mathematical background to understand the valuation concepts developed in Chapters 2 and 3. It provides an intuitive exposition of basic random

calculus. Concepts such as volatility and time, random walks, geometric Brownian motion, and Itô's lemma are exposed heuristically and given, where possible, an intuitive interpretation. This chapter also offers a few appetizers that we call paradoxes of finance: these paradoxes explain why forward exchange rates are biased predictors of future rates; why stock investing looks like a free lunch; and why success in portfolio management might have more to do with luck than with skill.

2. Chapter 2 develops generic pricing techniques for assets and derivatives. The chapter starts from basic concepts of utility theory and builds on these concepts to derive the notion of a stochastic discount factor, or pricing kernel. Pricing kernels are then used as the basis for the derivation of all subsequent pricing results, including the Black-Scholes/Merton model. We also show how pricing kernels relate to the hedging, or dynamic replication, approach that is the origin of all modern valuation principles. The chapter concludes with several applications to equity derivatives to demonstrate the power of the tools that are developed.
3. Chapter 3 specializes the pricing concepts of Chapter 3 to interest rate markets; namely bonds, swaps, and other interest rate derivatives. It starts with elementary concepts such as yield-to-maturity, zero-coupon rates, and forward rates; then moves on to naïve measures of interest rate risk such as duration and convexity and their underlying assumptions. An overview of interest rate derivatives precedes pricing models for interest rate instruments. These models fall into two conventional families: factor models, to which the notion of price of risk is central, and term-structure-consistent models, which are partial equilibrium models of derivatives pricing. The chapter ends with an interpretation of interest rates as options.
4. Chapter 4 is an expansion of the mathematical results in Chapter 1. It deals with a variety of mathematical topics that underlie derivatives pricing and portfolio allocation decisions. It describes in some detail random processes such as random walks, arithmetic and geometric Brownian motion, mean-reverting processes and jump processes. This chapter also includes an exposition of the rules of Itô calculus and contrasts it with the

competing Stratonovitch calculus. Related tools of stochastic calculus such as Kolmogorov equations and martingales are also discussed. The last two sections elaborate on techniques widely used to solve portfolio choice and option pricing problems: dynamic programming and partial differential equations.

We think that one virtue of the book is that the chapters are largely independent. Chapter 1 is essential to the understanding of the continuous-time sections in Chapters 2 and 3. Chapter 4 may be read independently, though previous chapters illuminate the concepts developed in each chapter much more completely.

Why Chapter 4 is at the end and not the beginning of this book is an almost aesthetic undertaking: Some finance experts think of mathematics as a way to learn finance. Our point of view is different. We feel that the joy of learning is in the process and not in the outcome. We also feel that finance can be a great way to learn mathematics.

1

Preliminary Mathematics

This chapter presents a brief overview of the technical language of modern finance. In the apparatus that we shall use, expressions such as *Random Walk*, *Brownian Motion*, and *Itô Calculus* may carry a shroud of mystery in the readers' minds. In an attempt to lift this shroud, we will be guilty of oversimplification and make no apology for that. Topics discussed in this chapter will be revisited and fleshed out in more detail in Chapter 4.

1.1 RANDOM WALK

Picture a particle moving on a line. Define X_t as the position of the particle at time t, with $X_0 = 0$. The particle moves one step forward (+1) or backward (−1) with equal probability at each instant of time, and successive steps are independent. At $t = 1$

$$\Pr[X_1 = -1] = \Pr[X_1 = 1] = 1/2$$

See Figure 1.1.

Similarly at $t = 2$, the particle can be at the positions −2, 0, 2 with probabilities 1/4, 1/2, and 1/4 respectively (see Figure 1.2). At $t = 3$, the values for X and their respective probabilities are

x	$\Pr[X_3 = x]$
-3	1/8
-1	3/8
$+1$	3/8
$+3$	1/8

We can calculate the expected value, variance, and standard deviation for X_t as of time zero. For example:

$$\mathbb{E}[X_1] = \left(\tfrac{1}{2} \times (-1)\right) + \left(\tfrac{1}{2} \times 1\right) = 0$$

$$Var[X_1] = \mathbb{E}[X_1 - \mathbb{E}X_1]^2 = \left(\tfrac{1}{2} \times (-1-0)^2\right) + \left(\tfrac{1}{2} \times (1-0)^2\right) = 1$$

$$SD[X_1] = \sqrt{Var[X_1]} = 1$$

Similarly,

$$\mathbb{E}[X_2] = \left(\tfrac{1}{4} \times (-2)\right) + \left(\tfrac{1}{2} \times 0\right) + \left(\tfrac{1}{4} \times 2\right) = 0$$

$$\begin{aligned} Var[X_2] &= \mathbb{E}[X_2 - \mathbb{E}X_2]^2 \\ &= \tfrac{1}{4} \times (-2-0)^2 + \tfrac{1}{2}(0-0)^2 + \tfrac{1}{4} \times (2-0)^2 = 2 \end{aligned}$$

$$SD[X_2] = \sqrt{Var[X_2]} = \sqrt{2}$$

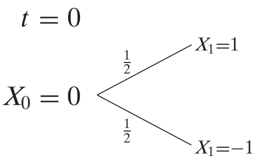

Figure 1.1

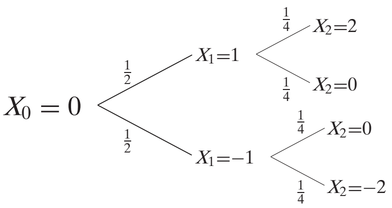

Figure 1.2

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Using a similar logic, we can find the expected value, variance, and standard deviation of X_t, conditional on $X_0 = 0$, for any $t > 0$:

	X_1	X_2	X_3	...	X_n
Expected Value $\mathbb{E}_0[X_t]$	0	0	0	...	0
Variance $Var_0[X_t]$	1	2	3		n
Standard Deviation $\sqrt{Var_0[X_t]}$	1	$\sqrt{2}$	$\sqrt{3}$		$\sqrt{n}$

We can generalize the above example. Now the variable X can go up a step with probability p or down a step with probability $q = 1 - p$. The step size is σ.

We can calculate the mean, variance, and standard deviation of X_1 as before. We now have

$$\mathbb{E}[X_1] = (p - q)\sigma = \mu$$

$$\mathbb{E}\left[X_1^2\right] = p\sigma^2 + q\sigma^2 = \sigma^2$$

$$Var[X_1] = \mathbb{E}X_1^2 - (\mathbb{E}X_1)^2 = 4\sigma^2 pq$$

$$SD[X_1] = \sqrt{Var[X_1]} = 2\sigma\sqrt{pq}$$

Define $\mu = (p - q)\sigma$. The variable μ is called the drift of X. X is said to follow a random walk with drift when $p \neq q$, and a driftless random walk when $p = q = 1/2$. In general we have

$$E[X_n] = n(p - q)\sigma = n\mu$$

$$Var[X_n] = 4\sigma^2 npq$$

$$SD[X_n] = \sqrt{Var[X_n]} = 2\sigma\sqrt{npq}$$

If the particle takes one step per unit of time then $n = t$, where t is the number of units of time. We see that the mean of a random walk is proportional to time, whereas the standard deviation is proportional to the square root of time. The latter result stems from the independence of the increments in a random walk. In a financial context, stock returns are often modeled as random walks. If $R_{t-1,t}$ represents the return on a stock between $t - 1$ and t, then the return over T periods is

$$R_{0,T} = R_{0,1} + R_{1,2} + \cdots + R_{T-1,T}$$

Returns in successive periods are assumed to be independent. This means that

$$\begin{aligned} Var\,[R_{0,T}] &= Var\,[R_{0,1} + R_{1,2} + \cdots + R_{T-1,T}] \\ &= Var\,[R_{0,1}] + Var\,[R_{1,2}] + \cdots + Var\,[R_{T-1,T}] \end{aligned}$$

Additionally, if the return in each period has a constant variance of σ^2, then

$$Var\,[R_{0,T}] = \sigma^2 T$$

and

$$SD[R_{0,T}] = \sigma\sqrt{T}$$

In finance, the standard deviation of a stock's returns is referred to as its *volatility*.

1.2 ANOTHER TAKE ON VOLATILITY AND TIME

We now give another perspective of volatility and time (Figure 1.3). X follows a two-dimensional random walk. The step size is σ. The angle θ_i at step i is random. After two steps, the distance D between the departure point X_0 and X_2 is given by

$$D^2 = (X_0C)^2 + (X_2C)^2$$

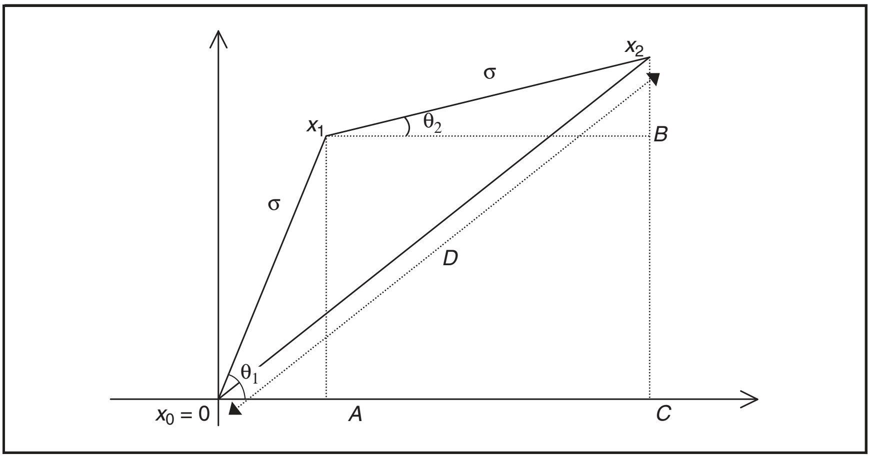

Figure 1.3

But

$$X_0C = X_0A + AC = X_0A + X_1B = \sigma \cos\theta_1 + \sigma \cos\theta_2$$

and

$$X_2C = X_2B + BC = X_2B + X_1A = \sigma \sin\theta_2 + \sigma \sin\theta_1$$

It follows that

$$\begin{aligned} D^2 &= \sigma^2[(\cos\theta_1 + \cos\theta_2)^2 + (\sin\theta_1 + \sin\theta_2)^2] \\ &= \sigma^2[\cos^2\theta_1 + \sin^2\theta_1 + \cos^2\theta_2 + \sin^2\theta_2 \\ &\quad + 2(\sin\theta_1 \sin\theta_2 + \cos\theta_1 \cos\theta_2)] \end{aligned}$$

Recall the equalities

$$\cos^2\theta_1 + \sin^2\theta_1 = 1$$
$$\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 = \cos(\theta_1 - \theta_2)$$

Using these equalities yields

$$D^2 = \sigma^2[2 + 2\cos(\theta_1 - \theta_2)]$$

Because the cosine term equals zero on average, we get

$$E(D^2) = 2\sigma^2$$
$$SD(D) = \sigma\sqrt{2}$$

1.3 A FIRST GLANCE AT ITÔ'S LEMMA

Recall the experiment discussed in the previous section (see Figure 1.4). It is easy to check that if we represent the process for X as shown in Figure 1.5, where $\mu \equiv (p - q)\sigma$, we get the same drift and volatility for each process.

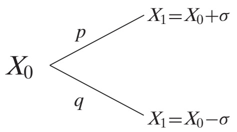

Figure 1.4

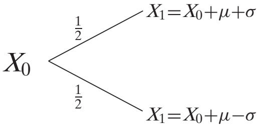

Figure 1.5

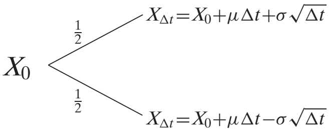

Figure 1.6

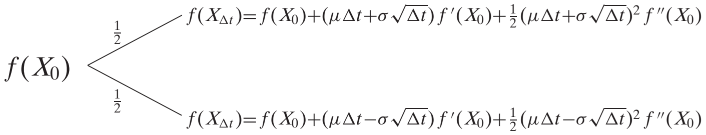

Figure 1.7

The second process can be expressed as

$$X_1 = X_0 + \mu \pm \sigma$$

or

$$\Delta X = \mu + \sigma\varepsilon$$

where $\Delta X = X_1 - X_0$ and $\varepsilon = 1$ or -1 with probability 1/2 for each outcome.

We now represent the binomial tree for a change in X given a time interval Δt (see Figure 1.6). The process for the binomial tree can be written as

$$\Delta X = X_{\Delta t} - X_0 = \mu\Delta t + \sigma\varepsilon\sqrt{\Delta t}$$

where the random variable ε keeps the same properties as above. Now, how can the variation on a function $f(X)$ be expressed? This is the question solved by Itô's lemma. A simple Taylor expansion to the second order gives the result shown in Figure 1.7.

For a small Δt, we may choose to neglect terms in $(\Delta t)^n$ (with $n > 1$) to get the outcome shown in Figure 1.8. In shorthand notation,