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## Abstract Regular Polytopes

Abstract regular polytopes stand at the end of more than two millennia of geometrical research, which began with regular polygons and polyhedra. They are highly symmetric combinatorial structures with distinctive geometric, algebraic, or topological properties, in many ways more fascinating than traditional regular polytopes and tessellations. The rapid development of the subject in the past 20 years has resulted in a rich new theory, featuring an attractive interplay of mathematical areas, including geometry, combinatorics, group theory, and topology. Abstract regular polytopes and their groups provide an appealing new approach to understanding geometric and combinatorial symmetry.

This is the first comprehensive up-to-date account of the subject and its ramifications, and meets a critical need for such a text, because no book has been published in this area of classical and modern discrete geometry since Coxeter's *Regular Polytopes* (1948) and *Regular Complex Polytopes* (1974). The book should be of interest to researchers and graduate students in discrete geometry, combinatorics, and group theory.

Peter McMullen is Professor of Mathematics at University College London.

Egon Schulte is Professor of Mathematics at Northeastern University in Boston.

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Volume 92

Abstract Regular Polytopes

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PETER McMULLEN  
*University College London*

EGON SCHULTE  
*Northeastern University*



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*To Donald Coxeter, a constant inspiration*

## Contents

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Preface	<i>page</i> xiii
<b>1 Classical Regular Polytopes</b>	1
1A The Historical Background	1
1B Regular Convex Polytopes	7
1C Extensions of Regularity	15
1D Regular Maps	17
<b>2 Regular Polytopes</b>	21
2A Abstract Polytopes	22
2B Regular Polytopes	31
2C Order Complexes	39
2D Quotients	42
2E C-Groups	49
2F Presentations of Polytopes	60
<b>3 Coxeter Groups</b>	64
3A The Canonical Representation	64
3B Groups of Spherical or Euclidean Type	71
3C Groups of Hyperbolic Type	76
3D The Universal Polytopes $\{p_1, \dots, p_{n-1}\}$	78
3E The Order of a Finite Coxeter Group	83
<b>4 Amalgamation</b>	95
4A Amalgamation of Polytopes	96
4B The Classification Problem	101
4C Finite Quotients of Universal Polytopes	103
4D Free Extensions of Regular Polytopes	106
4E Flat Polytopes and the FAP	109
4F Flat Polytopes and Amalgamation	115

<b>5 Realizations</b>	121
5A Realizations in General	121
5B The Finite Case	127
5C Apeirotopes	140
<b>6 Regular Polytopes on Space-Forms</b>	148
6A Space-Forms	148
6B Locally Spherical Polytopes	152
6C Projective Regular Polytopes	162
6D The Cubic Toroids	165
6E The Other Toroids	170
6F Relationships Among Toroids	172
6G Other Euclidean Space-Forms	175
6H Chiral Toroids	177
6J Hyperbolic Space-Forms	178
<b>7 Mixing</b>	183
7A General Mixing	183
7B Operations on Regular Polyhedra	192
7C Cuts	201
7D The Classical Star-Polytopes	206
7E Three-Dimensional Polyhedra	217
7F Three-Dimensional 4-Apeirotopes	236
<b>8 Twisting</b>	244
8A Twisting Operations	244
8B The Polytopes $\mathcal{L}^{\mathcal{K},\mathcal{G}}$	247
8C The Polytopes $2^{\mathcal{K}}$ and $2^{\mathcal{K},\mathcal{G}(s)}$	255
8D Realizations of $2^{\mathcal{K}}$ and $2^{\mathcal{K},\mathcal{G}(s)}$	259
8E A Universality Property of $\mathcal{L}^{\mathcal{K},\mathcal{G}}$	264
8F Polytopes with Small Faces	272
<b>9 Unitary Groups and Hermitian Forms</b>	289
9A Unitary Reflexion Groups	290
9B Hermitian Forms and Reflexions	298
9C General Considerations	305
9D Generalized Triangle Groups	320
9E Tetrahedral Diagrams	332
9F Circuit Diagrams with Tails	347
9G Abstract Groups and Diagrams	355
<b>10 Locally Toroidal 4-Polytopes: I</b>	360
10A Grünbaum's Problem	360
10B The Type $\{4,4,3\}$	363
10C The Type $\{4,4,4\}$	369
10D Cuts for the Types $\{4, 4, r\}$	378
10E Relationships Among Polytopes of Type $\{4, 4, r\}$	383

<b>11 Locally Toroidal 4-Polytopes: II</b>	387
11A The Basic Enumeration Technique	387
11B The Polytopes ${}_p\mathcal{T}_{(s,0)}^4 := \{\{6, 3\}_{(s,0)}, \{3, p\}\}$	392
11C Polytopes with Facets $\{6, 3\}_{(s,s)}$	400
11D The Polytopes ${}_6\mathcal{T}_{(s,0),(t,0)}^4 := \{\{6, 3\}_{(s,0)}, \{3, 6\}_{(t,0)}\}$	410
11E The Type $\{3, 6, 3\}$	417
11F Cuts of Polytopes of Type $\{6, 3, p\}$ or $\{3, 6, 3\}$	423
11G Hyperbolic Honeycombs in $\mathbb{H}^3$	431
11H Relationships Among Polytopes of Types $\{6, 3, p\}$ or $\{3, 6, 3\}$	437
<b>12 Higher Toroidal Polytopes</b>	445
12A Hyperbolic Honeycombs in $\mathbb{H}^4$ and $\mathbb{H}^5$	445
12B Polytopes of Rank 5	450
12C Polytopes of Rank 6: Type $\{3, 3, 3, 4, 3\}$	459
12D Polytopes of Rank 6: Type $\{3, 3, 4, 3, 3\}$	462
12E Polytopes of Rank 6: Type $\{3, 4, 3, 3, 4\}$	465
<b>13 Regular Polytopes Related to Linear Groups</b>	471
13A Regular Polyhedra	471
13B Connexions Among the Polyhedra	478
13C Realizations of the Polyhedra	484
13D The 4-Polytopes	490
13E Connexions Among 4-Polytopes	500
<b>14 Miscellaneous Classes of Regular Polytopes</b>	502
14A Locally Projective Regular Polytopes	502
14B Mixed Topological Types	509
Bibliography	519
Indices	
List of Symbols	539
Author Index	543
Subject Index	544



## Preface

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Regular polyhedra have been with us since before recorded history; the appeal of the beauty of geometric figures to the artistic senses well predates any mathematical investigation of them. However, it also seems to be the case that formal mathematics begins with such regular figures as an important topic, and a strong strain of mathematics since classical times has centred on them. Indeed, the subject of regular polyhedra has shown an enormous potential for revival. Before the present time, the most recent renaissance began in the early nineteenth century. The modern abstract theory started as an offshoot of this, but with the parallel (but separate) growth of the idea of geometries has taken on a new and vigorous life.

When we embarked in 1988 on the project of composing a coherent account of this modern theory, we had heard the old adage to the effect that, if one tries to write a paper, then one believes that everything is known, but when one starts to write a book, then one realizes that nothing is known. At the time, neither of us fully appreciated the truth of the saying. We began with a number of particular results, and an initial focus on a problem of classifying a certain class of abstract regular polytopes raised by Branko Grünbaum, but with only the merest outlines of a general theory. The foundations of the latter had been laid by Danzer and Schulte, although earlier intimations of what would be needed had appeared elsewhere (we elaborate on this at the beginning of Chapter 2). But the full development of the theory and its ramifications has occurred during the writing of this book, and has been largely inspired by it. Indeed, one major problem for us was that the subject seemed to be expanding faster than we could put it down on paper!

We have received much support from friends and colleagues during the preparation of this volume; as well, we have probably sorely tried their patience, because they had quite reasonably expected the book to have been published long before now. We wish particularly to thank Asia Ivič Weiss and Barry Monson for their encouragement and (we are sure) forbearance, but especially for providing various opportunities for us to collaborate. We are also very grateful to the NSF and NSA, which have supported Schulte's research, and have enabled McMullen to visit him. Above all, Donald Coxeter, the doyen of the classical subject, has been a constant inspiration to us.