Abstract Regular Polytopes

Abstract regular polytopes stand at the end of more than two millennia of geometrical research, which began with regular polygons and polyhedra. They are highly symmetric combinatorial structures with distinctive geometric, algebraic, or topological properties, in many ways more fascinating than traditional regular polytopes and tessellations. The rapid development of the subject in the past 20 years has resulted in a rich new theory, featuring an attractive interplay of mathematical areas, including geometry, combinatorics, group theory, and topology. Abstract regular polytopes and their groups provide an appealing new approach to understanding geometric and combinatorial symmetry.

This is the first comprehensive up-to-date account of the subject and its ramifications, and meets a critical need for such a text, because no book has been published in this area of classical and modern discrete geometry since Coxeter's *Regular Polytopes* (1948) and *Regular Complex Polytopes* (1974). The book should be of interest to researchers and graduate students in discrete geometry, combinatorics, and group theory.

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To Donald Coxeter, a constant inspiration

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Preface

Regular polyhedra have been with us since before recorded history; the appeal of the beauty of geometric figures to the artistic senses well predates any mathematical investigation of them. However, it also seems to be the case that formal mathematics begins with such regular figures as an important topic, and a strong strain of mathematics since classical times has centred on them. Indeed, the subject of regular polyhedra has shown an enormous potential for revival. Before the present time, the most recent renaissance began in the early nineteenth century. The modern abstract theory started as an offshoot of this, but with the parallel (but separate) growth of the idea of geometries has taken on a new and vigorous life.

When we embarked in 1988 on the project of composing a coherent account of this modern theory, we had heard the old adage to the effect that, if one tries to write a paper, then one believes that everything is known, but when one starts to write a book, then one realizes that nothing is known. At the time, neither of us fully appreciated the truth of the saying. We began with a number of particular results, and an initial focus on a problem of classifying a certain class of abstract regular polytopes raised by Branko Grünbaum, but with only the merest outlines of a general theory. The foundations of the latter had been laid by Danzer and Schulte, although earlier intimations of what would be needed had appeared elsewhere (we elaborate on this at the beginning of Chapter 2). But the full development of the theory and its ramifications has occurred during the writing of this book, and has been largely inspired by it. Indeed, one major problem for us was that the subject seemed to be expanding faster than we could put it down on paper!

We have received much support from friends and colleagues during the preparation of this volume; as well, we have probably sorely tried their patience, because they had quite reasonably expected the book to have been published long before now. We wish particularly to thank Asia Ivič Weiss and Barry Monson for their encouragement and (we are sure) forebearance, but especially for providing various opportunities for us to collaborate. We are also very grateful to the NSF and NSA, which have supported Schulte's research, and have enabled McMullen to visit him. Above all, Donald Coxeter, the doyen of the classical subject, has been a constant inspiration to us.