1

Classical Regular Polytopes

Our purpose in this introductory chapter is to set the scene for the rest of the book. We shall do this by briefly tracing the historical development of the subject. There are two main reasons for this. First, we wish to recall the historical traditions which lie behind current research. It is all too easy to lose track of the past, and it is as true in mathematics as in anything else that those who forget history may be compelled to repeat it. But perhaps more important is the need to base what we do firmly in the historical tradition. A tendency in mathematics to greater and greater abstractness should not lead us to abandon our roots. In studying abstract regular polytopes, we shall always bear in mind the geometric origins of the subject. We hope that this introductory survey will help the reader to find a firm basis from which to view the modern subject.

The chapter has four sections. In the first, we provide an historical sketch, leading up to the point at which the formal material of Chapter 2 begins. The second is devoted to an outline of the theory of regular convex polytopes, which provide so much of the motivation for the abstract definitions which we subsequently adopt. In the third, we treat various generalizations of regular polytopes, mainly in ordinary euclidean space, including the classical regular star-polytopes. In the fourth, we introduce regular maps, which are the first examples of abstract regular polytopes, although the examples considered here occur before the general theory was formulated.

1A The Historical Background

Regular polytopes emerge only gradually out of the mists of history. Apart from certain planar figures, such as squares and triangles, the cube, in the form of a die, was probably the earliest known to man. Gamblers would have used dice from the earliest days, and a labelled example helped linguists to work out the Etruscan words for "one" up to "six". The Etruscans also had dodecahedral dice; examples date from before 500BCE, and may even be much earlier. The other three "platonic" solids appear not to have been employed in gambling; two out of the three do not roll well in any case.

However, it is only somewhat later that the regular solids were studied for their own sakes, and the leap from them to the regular star-polyhedra, analogous to that from pentagon to pentagram, had to await the later middle ages. The nineteenth century first saw regular polytopes of higher dimension, but the real flowering of what is, in origin, one of the oldest branches of mathematics occurred only in the twentieth century.

In this section, we shall give a brief outline of the historical background to the theory of regular polytopes. This is not intended to be totally comprehensive, although we have attempted to give the salient features of more than two millenia of investigations in the subject.

The Classical Period

Before the Greeks

As we have already said, the cube was probably the first known regular polyhedron; certainly it was well known before the ideas of geometry and symmetry had themselves been formulated. Curiously, though, stone balls incised in ways that illustrate all the symmetry groups of the regular polyhedra were discovered in Scotland in the nineteenth century; they appear to date from the first half of the third millenium BCE (see [137, Chapter 7]).

The Egyptians were also aware of the regular tetrahedron and octahedron. As an aside, we pose the following question. Many attempts have been made to explain why the pyramids are the shape they are, or, more specifically, why the ratio of height to base of a square pyramid is always roughly the same particular number. In particular, such explanations often manage to involve π in some practical way, such as measurements by rolling barrels – the Egyptians' theoretical value $\frac{256}{81} \approx 3.16$ for π was fairly accurate. Is it possible, though, that a pyramid was intended to be half an octahedron? The actual angles of slopes of the four proper pyramids at Giza vary between $50^\circ 47'$ and $54^\circ 14'$; the last is only a little short of half the dihedral angle of the octahedron, namely, $\arccos(1/\sqrt{3}) \approx 54^\circ 44'$.

The Early Greeks

Despite various recent claims to the contrary, it seems clear that the Greeks were the first to conceive of mathematics as we now understand it. (The mere listing of, for example, Pythagorean triples does not constitute mathematics; a proof of Pythagoras's theorem obviously does.) According to Proclus (412–485CE), the discovery of the five regular solids was attributed by Eudemus to Pythagoras (c582–c500BCE) himself; the fact that a point of the plane is exactly surrounded by six equilateral triangles, four squares or three regular hexagons (these giving rise to the three regular tilings of the plane) was also known to his followers. They knew as well of the regular pentagram, apparently regarding it as a symbol of health; it has been suggested that this also gave them their first example of incommensurability. (For a fuller account of the origin of the regular solids, consult [438].)

The regular solids in ordinary space were named after Plato (Aristocles son of Ariston, 427–347BCE) by Heron; this seems to be one of the earliest mathematical misattributions. Indeed, their first rigorous mathematical treatment was by Theaetetus (c415–369BCE, when he was killed in battle). In his *Timaeus*, Plato does discuss the regular solids, but while his enthusiasm for and appreciation of the figures are obvious, it is also evident that his discussion falls short of a full mathematical investigation. However, one very perceptive idea does appear there. An equilateral triangle is regarded by him as formed from the six right-angled subtriangles into which it is split by its altitudes. The three solids with triangular faces (tetrahedron, octahedron, and icosahedron) are then built up from these subdivided triangles. This anticipates the construction of the Coxeter kaleidoscope of their reflexion planes by more than two millennia. But the general principle was not fully recognized by Plato; this is exhibited by his splitting of the square faces of the cube into four isosceles (instead of eight) triangles. Moreover, the dodecahedron is not seen in this way at all. In the *Timaeus*, one has the impression that the existence of the dodecahedron (identified with the universe) almost embarrasses Plato. The other four regular solids are identified with the four basic elements – tetrahedron = fire, octahedron = air, cube = earth, and icosahedron = water – in a preassumed scheme which is not at all scientific. (In the *Phaido*, amusingly, Plato also describes dodecahedra; they appear as stuffed leather balls made out of twelve multicoloured pentagonal pieces, an interesting near anticipation of some modern association footballs.)

To Plato's pupil Aristotle (384–322BCE) is attributed the mistaken assertion that the regular tetrahedron tiles ordinary space. Unfortunately, such was the high regard in which Aristotle was held in later times that his opinion was not challenged until comparatively recently, although its falsity could have been established at the time it was made.

Euclid

Euclid's *Elements* ($\Sigma\tau o\iota\chi\epsilon\iota\alpha$) is undoubtedly the earliest surviving true mathematics book, in the sense that it fully recognizes the characteristic mathematical paradigm of axiom–definition–theorem–proof. Until early in the twentieth century, parts of it, notably the first six Books ($\Sigma\chi o\lambda\iota\alpha$), provided, essentially unchanged, a fine introduction to basic geometry. It is unclear to what extent Euclid discovered his material or merely compiled it; our ignorance of Euclid himself extends to our being uncertain of more than that, as we are told by Proclus, he lived and worked in Alexandria at the time of Ptolemy I Soter (reigned 323–283BCE).

Of course, it is to Euclid that we look for the first rigorous account of the five regular solids; Proclus even claimed that *Elements* is designed to lead up to the discussion of them. Whether or not that contention can be justified, Book XIII is devoted to the regular solids. (Incidentally, this book and Book X are less than thoroughly integrated into the rest of the text, suggesting that they were incorporated from an already existing work, which may well have been written by Theaetetus himself.) The scholium (theorem) of that book demonstrates that there are indeed only five regular solids. The proof is straightforward, and (in essence) remains that still used: the angle at a vertex of a regular p-gon is $(1 - \frac{2}{p})\pi$, and so for q of them to fit around a vertex of a regular solid, one requires that $q(1-\frac{2}{p})\pi < 2\pi$, or, in other words,

1A1 $$\frac{1}{p}+\frac{1}{q}>\frac{1}{2}.$$

Euclid did not phrase the result in quite this way, but this is what the proof amounts to.

Further results about the regular solids occur in Books XIV and XV (which were written around 300CE, and so were not by Euclid), such as their metrical properties, and in particular some anticipation of duality. The details, and modern explanations of the results, are described in [120].

Archimedes

We should also briefly mention here a contribution of Archimedes (c287–212BCE). The Archimedean polyhedra themselves are beyond the scope of this book. However, Archimedes did use regular polygons – actually, the 96-gon – to find his famous bounds $3\frac{10}{71} < \pi < 3\frac{1}{7}$. The remainder of the many mathematical results of Archimedes are not relevant to the topic of this book, but their significance cannot be allowed to pass altogether unnoticed.

The Romans

The Romans were fine architects and engineers, but contributed less to mathematics. However, among the various works attributed to the great pagan philosopher Anicius Manlius Severinus Boetius (c480–524CE) (his name is usually miswritten as “Boethius”) is a translation of most of the first three books of Euclid. Since he certainly translated Plato and Aristotle (with a view to reconciling them), this is a possibility which cannot lightly be dismissed.

The Mediaeval Period

The Early Middle Ages

The Christian Roman Empire got off to a poor start in its treatment of learning; under a decree of Emperor Theodosius I (“the Great”) concerning pagan monuments, in around 389–391 Bishop Theophilus ordered that the great library of Alexandria (or, at least, that part in the Serapeum) be pillaged. (It is uncertain how much of the original library had survived to this time; it is said – though the event is disputed – that the larger part, the Brucheum, was burnt around 47BCE when Caesar set fire to the Egyptian fleet during the Roman civil wars. The later story of the Muslim destruction under ’Amr is of much more dubious provenance.) A little later, the last of the Alexandrine philosophers, the talented and beautiful Hypatia (c375–415), was flayed with oyster shells by a Christian mob at the instigation of Bishop Cyril (who was later canonized).

The attitude of the Byzantine (Eastern Roman) Empire to mathematics (and the other sciences) was distinctly ambiguous, alternating between encouragement and suppression. Justinian I (reigned 527–565) initially seemed supportive, but soon closed the

Academy at Athens in 529, although there was probably little resulting loss to mathematics. It is due to a few people in the ninth (particularly Leo "the mathematician") and tenth centuries that we have the Greek manuscripts of Euclid which survive; the earliest dates from 888. Similarly, a tenth-century manuscript of Archimedes was re-used in the twelfth century (a palimpsest) for religious texts; fortunately, the original mathematics can be recovered by modern techniques. It must be concluded that the Byzantines preserved rather than added to the corpus of knowledge.

The mathematical torch was also carried on by the Arabs, but they too seem to have added little to geometry, although their translation of Euclid helped to preserve it. (The contributions of the Islamic world to algebra are a quite different matter.) They had a good empirical knowledge of symmetry; in the Alhambra there are patterns which exemplify many of the seventeen possible planar symmetry groups (and the rest can be produced by slight modifications of some of the others).

The Later Middle Ages

From about the twelfth century, mathematical knowledge began seeping back into western Europe. Around the 1120s, Aethelard (Adelard) of Bath, known as "Philosophus Anglorum", produced a translation of Euclid; while he knew Greek, this is more likely to have been from Arabic. (It was first printed in Venice in 1482 under the name of Campanus of Novara, with an unhelpful commentary, but the attribution to Aethelard is universally accepted.)

Rather later, Thomas Bradwardine "the Profound Doctor" (c1290–1349), Archbishop of Canterbury for just forty days after his consecration (he died of plague), systematically investigated star-polygons, obtaining $\{\frac{n}{d}\}$ by stellating $\{\frac{n}{d-1}\}$. (The notation will be explained later in the chapter.)

Although Kepler and Poinsot (see the following subsection) are credited with discovering the regular star-polyhedra in three dimensions, the polyhedron $\{\frac{5}{2}, 5\}$ was depicted in 1420 by Paolo Uccello (1397–1475), while $\{5, \frac{5}{2}\}$ occurs in an engraving of 1568 by Wenzel Jamnitzer (1508–85); however, it is unlikely that they fully appreciated the differences between these figures and others that they drew.

The Modern Period

Before Schläfli

Johannes Kepler (1571–1630) began the modern investigation of regular polytopes by his discovery of the two star-polyhedra $\{\frac{5}{2}, 5\}$ (strictly, perhaps, a rediscovery) and $\{\frac{5}{2}, 3\}$ (see [248, p. 122]). He also investigated various regular star-polygons, particularly the heptagons; for the latter, he showed that the side lengths of the three heptagons $\{7\}$, $\{\frac{7}{2}\}$ and $\{\frac{7}{3}\}$ inscribed in the unit circle are the roots of the equation

1A2 $$\lambda^6 - 7\lambda^4 + 14\lambda^2 - 7 = 0.$$

In a sense, Kepler stands on a cusp. The lingering effect of mediaeval (or perhaps even classical) thought on him shows in his attempt to relate the relative sizes of the orbits of the planets to the ratios of in- and circumradii of the regular polyhedra; later in his life he demonstrated that these orbits were ellipses.

The Greeks had proved that certain regular polygons, notably the pentagon, were constructible using ruler and compass alone. In 1796, the young Carl Friedrich Gauss (1777–1855) showed that, if a regular n-gon $\{n\}$ can be so constructed, then n is a power of 2 times a product of distinct Fermat primes, of the form

1A3 $$p = 2^{2^k} + 1$$

for some k; in fact, the condition is sufficient as well as necessary. The only known Fermat primes are those for $k = 0, 1, 2, 3, 4$; if there are no others, then an odd such n is a divisor of $2^{32} - 1$. In 1809, Louis Poinsot (1777–1859) rediscovered the first two regular star-polyhedra, and found their duals $\{5, \frac{5}{2}\}$ (again, perhaps really a rediscovery) and $\{3, \frac{5}{2}\}$ (see [342]); very soon afterwards, in 1811, Augustin Louis Cauchy (1789–1857) proved that the list of such regular star-polyhedra was now complete (see [76]).

Schläfli

At a time when very few mathematicians had any concept of working in higher dimensional spaces, Ludwig Schläfli (1814–95) discovered regular polytopes and honeycombs in four and more dimensions around 1850 (see [355, §17, 18]). In fact, he found all the groups of the regular polytopes whose symmetry groups are generated by reflexions in hyperplanes in euclidean spaces. But against all his evidence he refused to recognize the dual pair $\{\frac{5}{2}, 5\}$ and $\{5, \frac{5}{2}\}$ as "genuine" polyhedra (because they have non-zero genus), and so would not accept either the regular 4-polytopes which have these as facets or vertex-figures, even though calculating the spherical volumes of corresponding tetrahedra on the 3-sphere was a central part of his treatment.

From 1880 onwards, the regular polytopes in higher dimensions were rediscovered many times, beginning with Stringham [405]. We refer to [120, p. 144] for the relevant details. Edmund Hess [215] found the remaining regular star-polytopes, and S. L. van Oss [337] proved that the enumeration was complete. (For an argument avoiding consideration of each separate case, see [280] and Section 7D in this work.)

Coxeter

The subject of regular polytopes had gone into somewhat of a decline when it was taken up by H. S. M. (Donald) Coxeter (born 1907). His investigations and consolidation of the theory culminated in his famous book *Regular Polytopes* [120], whose first edition was published in 1948. His contributions are too numerous to list here individually, but we should at least mention Coxeter diagrams and the complete classification of the discrete euclidean reflexion groups among all Coxeter groups. We shall mention this latter material in Section 1B.

But Coxeter also pointed towards later developments of the theory. In particular, when J. F. Petrie (1907–72) (the inventor of the skew polygon which bears his name) found the two regular skew apeirohedra $\{4, 6 \mid 4\}$ and $\{6, 4 \mid 4\}$, he immediately found the third $\{6, 6 \mid 3\}$, and set the whole theory in a general context (see [105]). He also looked at regular maps and their automorphism groups, regarding the star-polyhedra as particular examples; he first observed that the Petrie polygons of a regular map themselves (usually) form another regular map (see [131, p. 112]). We shall provide an introduction to this area in Section 1D.

In 1975, Grünbaum (see [198]) gave the theory a further impetus. He generalized the regular skew polyhedra, by allowing skew polygons as faces as well as vertex-figures. He found eight more individual examples and twelve infinite families (with non-congruent realizations of isomorphic apeirohedra), and Dress [148, 150] completed the classification by finding the final case and establishing the completeness of the list. Again, we shall consider this work later, in the appropriate place (see Section 7E).

Finally, regular polytopes also formed the cradle of Tits's work on buildings (see [415–417]). Buildings of spherical type are the natural geometric counterparts of simple Lie groups of Chevally type. Regular polytopes, or, more generally, Coxeter complexes (see Sections 2C and 3A), occur here as fundamental structural components, namely, as the "apartments" of buildings. In a further generalization, Buekenhout [55, 58] introduced the notion of a diagram geometry to find a geometric interpretation for the twenty-six sporadic groups (see [14, 143, 244]). Although we shall not discuss buildings and diagram geometries in detail, the present book has nevertheless been considerably influenced by these developments.

History teaches us that the subject of regular polyhedra has shown an enormous potential for revival. One natural explanation is that the beauty of the geometric figures appeals to the artistic senses [20, 384].

1B Regular Convex Polytopes

We begin this section with a short discussion of convexity, which we shall need again in Chapter 5. For fuller details, we refer the reader to any one of a number of standard texts, for example [197, 357].

A subset K of n-dimensional euclidean space $\mathbb{E}^n$ is *convex* if, with each two of its points x and y, it contains the *line segment*

$$[xy] := \{(1-\lambda)x + \lambda y \mid 0 \leqslant \lambda \leqslant 1\}.$$

The intersection of convex sets is again convex, and so the *convex hull* conv S of a set $S \subseteq \mathbb{E}^n$ is well defined as the smallest convex set which contains S. The convex hull of a finite set of points is a *convex polytope*; in this section, we shall frequently drop the qualifying term "convex" and talk simply about a polytope. A polytope P is *k-dimensional*, or a *k-polytope*, if its affine hull is k-dimensional. Here, an *affine subspace* of $\mathbb{E}^n$ is a subset A which contains each line

$$xy := \{(1-\lambda)x + \lambda y \mid \lambda \in \mathbb{R}\}$$

between two points $x, y \in A$; the *affine hull* aff S of a subset S is similarly the smallest affine subspace of $\mathbb{E}^n$ which contains S.

Bear in mind that a non-empty affine subspace A is a translate of a unique linear subspace

$$L := A - A = A - x$$

for any $x \in A$; by definition $\dim A := \dim L$. The empty set $\emptyset$ is the affine subspace of dimension -1; it is also a polytope. We further refer to 2-polytopes as *polygons* and to 3-polytopes as *polyhedra*.

The simplest example of an n-polytope is an n*-simplex*, which is the convex hull of an affinely independent set of $n + 1$ points. Here, a set $\{a_0, a_1, \ldots, a_n\}$ is *affinely independent* if, whenever $\lambda_0, \lambda_1, \ldots, \lambda_n \in \mathbb{R}$ are such that

$$\sum_{i=0}^{n} \lambda_i a_i = o, \qquad \sum_{i=0}^{n} \lambda_i = 0,$$

then $\lambda_0 = \lambda_1 = \cdots = \lambda_n = 0$; this is the natural extension of the notion of linear independence. (We use "o" to denote the zero vector.)

A hyperplane

$$H(u, \alpha) := \{x \in \mathbb{E}^n \mid \langle x, u \rangle = \alpha\}$$

supports a convex set K, with *outer normal* u, if

$$\alpha = \sup\{\langle x, u \rangle \mid x \in K\}.$$

The intersection $H(u, \alpha) \cap K$ is then an (*exposed*) *face* of K. An n-polytope P has faces of each dimension $0, \ldots, n - 1$, which are themselves polytopes. Often, $\emptyset$ and P itself are counted as faces of P, called the *improper* faces; the other faces are *proper*. We write $\mathcal{P}(P) = \mathcal{P}$ for the family of all faces of P. The faces of dimensions 0, 1, $n - 2$ and $n - 1$ are also referred to as its *vertices, edges, ridges* and *facets*, respectively; more generally, a face of dimension j is called a j*-face*.

The notation vert P is usual for the set of vertices of a polytope P; then $P = \operatorname{conv}(\operatorname{vert} P)$. If $v \in \operatorname{vert} P$, and if H is a hyperplane which strictly separates v from $\operatorname{vert} P \setminus \{v\}$, then $H \cap P$ is called a *vertex-figure* of P at v. In the cases we shall consider in the following, we may usually choose the vertices of the vertex-figure at v in some special way; traditionally, they are the midpoints of the edges through v, although we shall frequently violate the strict terms of the definition, and take the other vertices of the edges through v instead.

Before we proceed further, we list various properties of a convex n-polytope P, which will motivate many of the definitions we adopt in Chapter 2.

- $\mathcal{P}$ is a lattice, under the partial ordering $F \leqslant G$ if and only if $F \subseteq G$. The *meet* of two faces F and G is then $F \wedge G := F \cap G$, and the *join* $F \vee G$ is the (unique) smallest face of P which contains F and G.
- If $F < G$ are two faces of P with $\dim G - \dim F = 2$, then there are exactly two faces J of P such that $F < J < G$.

- For every two faces F, G of P with $F \leqslant G$, the *section*

$$G/F := \{J \in \mathcal{P} \mid F \leqslant J \leqslant G\}$$

of $\mathcal{P}$ is isomorphic to the face-lattice of a polytope of dimension $\dim G - \dim F - 1$. (For $F = \emptyset$, we have $G/F = G$ by a minor abuse of notation; when $\dim F \geqslant 0$, proceed by induction, namely, by successive construction of vertex-figures.)

Two faces F and G of P are called *incident* if $F \leqslant G$ or $G \leqslant F$.

- If $\dim P \geqslant 2$, then $\mathcal{P}$ is *connected*, in the sense that any two proper faces F and G of P can be joined by a chain $F =: F_0, F_1, \ldots, F_k := G$ of proper faces of P, such that F_{i-1} and F_i are incident for $i = 1, \ldots, k$. Hence, $\mathcal{P}$ is *strongly connected*, in that the same is true for every section G/F of $\mathcal{P}$ such that $\dim G \geqslant \dim F + 3$.
- The boundary bd P of P is homeomorphic to an $(n-1)$-sphere; in particular, if $n \geqslant 3$, then bd P is simply connected.

We call two polytopes P and Q (*combinatorially*) *isomorphic* if their face-lattices $\mathcal{P}(P)$ and $\mathcal{P}(Q)$ are isomorphic, so that there is a one-to-one inclusion preserving correspondence between the faces of P and those of Q. Similarly, P and Q are *dual* if $\mathcal{P}(P)$ and $\mathcal{P}(Q)$ are anti-isomorphic, giving a one-to-one inclusion reversing correspondence between the faces of P and those of Q. The notation P^* for a dual of P will occur quite often.

A *flag* of an n-polytope P is a maximal subset of pairwise incident faces of P; thus, it is of the form $\{F_{-1}, F_0, \ldots, F_{n-1}, F_n\}$, with

$$F_{-1} \subset F_0 \subset \cdots \subset F_{n-1} \subset F_n.$$

Here we introduce the conventions $F_{-1} := \emptyset$ and $F_n := P$ for an n-polytope P; the inclusions are strict, so that $\dim F_j = j$ for each $j = 0, \ldots, n-1$. The improper faces $\emptyset$ and P are often omitted from the specification of a flag, since they belong to all of them. The family of flags of P is denoted $\mathcal{F}(P)$. Flags thus have the following properties.

- For each $j = 0, \ldots, n-1$, there is a unique flag $\Phi^j \in \mathcal{F}(P)$ which differs from a given flag Φ in its j-face alone. Two such flags Φ and Φ^j are called *adjacent*, or, more exactly, *j-adjacent*.
- P is *strongly flag-connected*, in that for each two flags Φ and Ψ of P, there exists a chain $\Phi =: \Phi_0, \Phi_1, \ldots, \Phi_k := \Psi$, such that Φ_{i-1} and Φ_i are adjacent for each $i = 1, \ldots, k$, and $\Phi \cap \Psi \subseteq \Phi_i$ for each $i = 1, \ldots, k-1$.

The *symmetry group* $G(P)$ of P consists of the isometries g of $\mathbb{E}^n$ such that $Pg = P$.[†] Then P is called *regular* if $G(P)$ is transitive on the family $\mathcal{F}(P)$ of flags of P; this form of the definition seems to have been given first by Du Val in [156, p. 63].

Alternative definitions of regularity of an n-polytope are common in the literature. We list some of them here; a comprehensive discussion of this topic occurs in [279].

[†] In such algebraic contexts, we write maps after their arguments throughout the book. Compositions of maps thus occur in their natural order; that is, they are read from left to right. Note that these conventions are a change from those in some of our earlier publications.

- A polygon is regular if its edges have the same length, and the angles at its vertices are equal (or, its vertices lie on a circle).
- For $n \geqslant 3$, an n-polytope is regular if its facets are regular and congruent (or isomorphic), and its vertex-figures are isomorphic. (This formulation depends on Cauchy's rigidity theorem; see [242, p. 335].)
- For every n, an n-polytope P is regular if, for each $j = 0, \ldots, n-1$, its symmetry group $G(P)$ is transitive on the j-faces of P.

A *reflexion* R in $\mathbb{E}^n$ is an involutory isometry; it has a *mirror*

$$\{x \in \mathbb{E}^n \mid xR = x\}$$

of fixed points with which it is identified, so that the same notation R is employed for it. A *hyperplane reflexion* has a hyperplane as its mirror.

A *Coxeter group* is one of the form $G := \langle R_0, \ldots, R_{n-1} \rangle$, the group generated by $R_0, \ldots, R_{n-1}$, which satisfies relations solely of the form

$$(R_i R_j)^{p_{ij}} = E,$$

the identity, where the $p_{ij} = p_{ji}$ are positive integers (or infinity) satisfying $p_{jj} = 1$ for each $j = 0, \ldots, n-1$. In addition, we call G a *string* (Coxeter) group if $p_{ij} = 2$ whenever $0 \leqslant i < j-1 \leqslant n-2$; this group is then denoted $[p_1, \ldots, p_{n-1}]$. We shall discuss Coxeter groups in full generality in Chapter 3.

1B1 Theorem *The symmetry group $G(P)$ of a regular convex n-polytope P is a finite string Coxeter group, with generators R_j for $j = 0, \ldots, n-1$ which are hyperplane reflexions, and $p_j := p_{j-1,j} \geqslant 3$ for $j = 1, \ldots, n-1$ (in the previous notation). Conversely, any finite string Coxeter group for which $p_j \geqslant 3$ for $j = 1, \ldots, n-1$ is the symmetry group of a regular convex polytope.*

Proof. Let us explain how this result arises. Fix a flag $\Phi = \{F_{-1}, F_0, \ldots, F_{n-1}, F_n\}$ of a regular n-polytope P, with the conventions introduced previously. Denote by q_j the centroid of F_j for $j = 0, \ldots, n$ (by this, we mean the centroid of its vertices), and, for each $j = 0, \ldots, n-1$, let

1B2 $$H_j := \operatorname{aff}\{q_i \mid i \neq j\}.$$

It is not hard to see that $\{q_0, \ldots, q_n\}$ is affinely independent, so that each H_j is a hyperplane. If R_j is the (hyperplane) reflexion whose mirror is H_j, then $G(P) = \langle R_0, \ldots, R_{n-1} \rangle$.

We see this as follows. In any n-polytope P, and for any flag Φ of P, for each $j = 0, \ldots, n-1$, let Φ^j (as before) be the unique flag of P which is j-adjacent to Φ. Then R_j is the unique symmetry of P which interchanges Φ and Φ^j. The simple-connectedness of the boundary of P (for $n \geqslant 3$ – the case $n = 2$ is trivial) then leads directly to the first assertion of the theorem. Many of the details of the proof are exactly as in that of Theorem 1B3, and so we shall postpone them until then.

We shall leave the converse of Theorem 1B1 until we have discussed Coxeter groups in more detail. However, the essence of the argument lies in the fact that a finite