PERIOD MAPPINGS AND PERIOD DOMAINS

The concept of a period of an elliptic integral goes back to the 18th century. Later Abel, Gauss, Jacobi, Legendre, Weierstrass, and others made a systematic study of these integrals. Rephrased in modern terminology, these give a way to encode how the complex structure of a two-torus varies, thereby showing that certain families contain all elliptic curves. Generalizing to higher dimensions resulted in the formulation of the celebrated Hodge conjecture, and in an attempt to solve this, Griffiths generalized the classical notion of period matrix and introduced period maps and period domains which reflect how the complex structure for higher dimensional varieties varies. The basic theory as developed by Griffiths is explained in the first part of this book. Then, in the second part spectral sequences and Koszul complexes are introduced and are used to derive results about cycles on higher dimensional algebraic varieties such as the Noether-Lefschetz theorem and Nori's theorem. Finally, in the third part differential geometric methods are explained, leading up to proofs of Arakelov-type theorems, the theorem of the fixed part, the rigidity theorem, and more. Higgs bundles and relations to harmonic maps are discussed, and this leads to striking results such as the fact that compact quotients of certain period domains can never admit a Kähler metric or that certain lattices in classical Lie groups can't occur as the fundamental group of a Kähler manifold.

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Phillip Griffiths

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More information

Preface

What to expect of this book: Our aim is to give an up-to-date exposition of the theory of period maps originally introduced by Griffiths. It is mainly intended as a textbook for graduate students. However, it should also be of interest to any mathematician wishing to get an introduction to those aspects of Hodge theory that are related to Griffiths' theory.

Prerequisites: We assume that the reader has encountered complex or complex algebraic manifolds before. We have in mind familiarity with the concepts from the first chapters of the book by Griffiths and Harris [103] or from the first half of Forster's book [79].

A second prerequisite is some familiarity with algebraic topology. For the fundamental group the reader may consult Forster's book [79]. Homology and cohomology are at the base of Hodge theory and so the reader should know either simplicial or singular homology and cohomology. A good source for the latter is Greenberg's book [98].

Next, some familiarity with basic concepts and ideas from differential geometry such as smooth manifolds, differential forms, connections, and characteristic classes is required. Apart from [103], the reader is invited to consult [41] and [110]. To have an idea what we actually use in the book, see the three appendices. We occasionally refer to these in the main body of the book. We particularly recommend that the reader do the problem sets which are meant to provide the techniques necessary to calculate all sorts of invariants for concrete examples in the main text.

Contents of the book: The concept of a period-integral goes back to the nineteenth century; it has been introduced by Legendre and Weierstrass for integrals of certain elliptic functions over closed circuits in the dissected complex plane and of course is related to periodic functions like the Weierstrass $\mathcal{P}$-function. In modern terminology we would say that these integrals describe exactly how the complex structure of an elliptic curve varies. From this point of view the analogous question for higher-genus curves becomes apparent and leads to period matrices and Torelli's theorem for curves. We have treated this historical starting point in the first chapter.

Because we introduce the major concepts of the book by means of examples, the first chapter can be viewed as a motivation for the rest of the book. Indeed period mappings and period domains appear in it, as well as several other important notions and ideas such as monodromy of a family, algebraic cycles, the Hodge decomposition, and the Hodge conjecture. This chapter is rather long because we also wanted to address several important aspects of the theory not treated in later chapters. Below we say more about this, but we pause here to point out that the nature of the first chapter makes it possible to use it entirely for a first course on period maps.

For instance, we introduce mixed Hodge theory in this chapter and explain the geometry behind it, but of course only in the simplest situations. We look at the cohomology of a singular curve on the one hand, and on the other hand we consider the limit mixed Hodge structure on the cohomology for a degenerating family of curves. This second example leads to the *asymptotic* study and becomes technically complicated in higher dimensions and falls beyond the modest scope of our book. Nevertheless it motivates certain results in the rest of the book such as those concerning variations of Hodge structure over the punctured disk (especially the monodromy theorem) which are considered in detail in Chapter 13.

The beautiful topic of Picard–Fuchs equations, treated in relation to a family of elliptic curves, is not discussed again in later chapters. We certainly could have done this, for instance after our discussion of the periods for families of hypersurfaces in projective space (Section 3.2). Lack of time and space prevented us from doing this. We refer the interested reader to [15] and [49] where some calculations are carried out that are significant for important examples occurring in mirror-symmetry and can be understood after reading the material in the first part.

The remainder of the first part of the book is devoted to fleshing out the ideas presented in the first chapter. Cohomology being essentially the only available invariant, we explain in Chapter 2 how the Kähler assumption implies that one can pass from the type decomposition on the level of complex forms to the level of cohomology classes. This is the Hodge decomposition. We show how to compute the Hodge decomposition in a host of basic examples. In Chapter 3, we pave the way for the introduction of the period map by looking at invariants related to cohomology that behave holomorphically (although this is shown much later, in Chapter 6, when we have developed the necessary tools). Griffiths' intermediate Jacobians and the Hodge (p, p)-classes are central in this chapter; we also calculate the Hodge decomposition of the cohomology of projective hypersurfaces in purely algebraic terms. This will enable us on various occasions to use these as examples to illustrate the theory. For instance, infinitesimal Torelli is proved for them in Chapter 5, Noether–Lefschetz-type theorems in Chapter 7, and variational Torelli theorems in Chapter 8.

In Chapter 4 the central concepts of this book finally can be defined after we have illustrated the role of the monodromy in the case of Lefschetz pencils. Abstract variations of Hodge structure then are introduced. In a subsequent chapter these are studied from an infinitesimal point of view.

In Part II spectral sequences are treated, and with these, previous loose ends are tied up. Another central tool, developed in Chapter 7, is the theory of Koszul complexes. Through Donagi's symmetrizer lemma and its variants, these turn out to be crucial for applications such as Noether–Lefschetz theorems and variational Torelli, which are treated in Chapters 7 and 8, respectively.

Then in Chapter 9 we turn to another important ingredient in the study of algebraic cycles, the normal functions. Their infinitesimal study leads to a proof of a by-now classical theorem due to Voisin and Green stating that the image of the Abel–Jacobi map for "very general" odd dimensional hypersurfaces of projective space is as small as it can be, at least if the degree is large enough.

We finish Part II with a sophisticated chapter on Nori's theorem, which has profound consequences for algebraic cycles, vastly generalizing pioneering results by Griffiths and Clemens.

In the final part of the book we turn to purely differential geometric aspects of period domains. Our main goal here is to explain in Chapter 13 those curvature properties that are relevant for period maps. Prior to that chapter, in Chapters 11 and 12, we present several more or less well known notions and techniques from differential geometry, which involve the Lie theory needed for period domains.

Among the various important applications of these basic curvature properties, we have chosen to prove in Chapter 13 the theorem of the fixed part, the rigidity theorem, and the monodromy theorem. We also show that the period map extends as a proper map over the locus where the local monodromy is finite, and we discuss some important consequences. In the same chapter we introduce Higgs bundles and briefly explain how these come up in Simpson's work on nonabelian Hodge theory.

In the final chapter we broaden our point of view in that we look more generally at harmonic and pluriharmonic maps with the target a locally symmetric space. Using the results of this study, we can, for instance, show that compact quotients of period domains of even weight are never homotopy equivalent to Kähler manifolds.

To facilitate reading, we start every chapter with a brief outline of its content. To encourage the reader to digest the considerable number of concepts and techniques we have included many examples and problems. For the more difficult problems we have given hints or references to the literature. Finally, we end every chapter with some historical remarks.

It is our pleasure to thank various people and institutions for their help in the writing of this book.

We are first of all greatly indebted to Phillip Griffiths who inspired us either directly or indirectly over all the years we have been active as mathematicians; through this book we hope to promote some of the exciting ideas and results related to cycles initiated by him and pursued by others, such as Herb Clemens, Mark Green, Madhav Nori, and Claire Voisin.

Special thanks go to Domingo Toledo for tremendous assistance with the last part of the book and to Jan Nagel who let us present part of his work in Chapter 10. Moreover, he and several others critically read first drafts of this book: Daniel Huybrechts, James Lewis, Jacob Murre, James Parson, Jens Piontkowski, Alexander Schwarzhaupt, and Eckart Viehweg; we extend our gratitude to all of them.

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