

## Contents

---

<i>Preface</i>	<i>page xi</i>
<b>1 Groups and permutations</b>	<b>1</b>
1.1 Introduction	1
1.2 Groups	2
1.3 Permutations of a finite set	6
1.4 The sign of a permutation	11
1.5 Permutations of an arbitrary set	15
<b>2 The real numbers</b>	<b>22</b>
2.1 The integers	22
2.2 The real numbers	26
2.3 Fields	27
2.4 Modular arithmetic	28
<b>3 The complex plane</b>	<b>31</b>
3.1 Complex numbers	31
3.2 Polar coordinates	36
3.3 Lines and circles	40
3.4 Isometries of the plane	41
3.5 Roots of unity	44
3.6 Cubic and quartic equations	46
3.7 The Fundamental Theorem of Algebra	48
<b>4 Vectors in three-dimensional space</b>	<b>52</b>
4.1 Vectors	52
4.2 The scalar product	55
4.3 The vector product	57
4.4 The scalar triple product	60

viii	<i>Contents</i>	
4.5	The vector triple product	62
4.6	Orientation and determinants	63
4.7	Applications to geometry	68
4.8	Vector equations	72
<b>5</b>	<b>Spherical geometry</b>	74
5.1	Spherical distance	74
5.2	Spherical trigonometry	75
5.3	Area on the sphere	77
5.4	Euler's formula	79
5.5	Regular polyhedra	83
5.6	General polyhedra	85
<b>6</b>	<b>Quaternions and isometries</b>	89
6.1	Isometries of Euclidean space	89
6.2	Quaternions	95
6.3	Reflections and rotations	99
<b>7</b>	<b>Vector spaces</b>	102
7.1	Vector spaces	102
7.2	Dimension	106
7.3	Subspaces	111
7.4	The direct sum of two subspaces	115
7.5	Linear difference equations	118
7.6	The vector space of polynomials	120
7.7	Linear transformations	124
7.8	The kernel of a linear transformation	127
7.9	Isomorphisms	130
7.10	The space of linear maps	132
<b>8</b>	<b>Linear equations</b>	135
8.1	Hyperplanes	135
8.2	Homogeneous linear equations	136
8.3	Row rank and column rank	139
8.4	Inhomogeneous linear equations	141
8.5	Determinants and linear equations	143
8.6	Determinants	144
<b>9</b>	<b>Matrices</b>	149
9.1	The vector space of matrices	149
9.2	A matrix as a linear transformation	154
9.3	The matrix of a linear transformation	158

*Contents*

ix

9.4	Inverse maps and matrices	163
9.5	Change of bases	167
9.6	The resultant of two polynomials	170
9.7	The number of surjections	173
<b>10</b>	<b>Eigenvectors</b>	175
10.1	Eigenvalues and eigenvectors	175
10.2	Eigenvalues and matrices	180
10.3	Diagonalizable matrices	184
10.4	The Cayley–Hamilton theorem	189
10.5	Invariant planes	193
<b>11</b>	<b>Linear maps of Euclidean space</b>	197
11.1	Distance in Euclidean space	197
11.2	Orthogonal maps	198
11.3	Isometries of Euclidean $n$ -space	204
11.4	Symmetric matrices	206
11.5	The field axioms	211
11.6	Vector products in higher dimensions	212
<b>12</b>	<b>Groups</b>	215
12.1	Groups	215
12.2	Subgroups and cosets	218
12.3	Lagrange’s theorem	223
12.4	Isomorphisms	225
12.5	Cyclic groups	230
12.6	Applications to arithmetic	232
12.7	Product groups	235
12.8	Dihedral groups	237
12.9	Groups of small order	240
12.10	Conjugation	242
12.11	Homomorphisms	246
12.12	Quotient groups	249
<b>13</b>	<b>Möbius transformations</b>	254
13.1	Möbius transformations	254
13.2	Fixed points and uniqueness	259
13.3	Circles and lines	261
13.4	Cross-ratios	265
13.5	Möbius maps and permutations	268
13.6	Complex lines	271
13.7	Fixed points and eigenvectors	273

13.8	A geometric view of infinity	276
13.9	Rotations of the sphere	279
<b>14</b>	<b>Group actions</b>	<b>284</b>
14.1	Groups of permutations	284
14.2	Symmetries of a regular polyhedron	290
14.3	Finite rotation groups in space	295
14.4	Groups of isometries of the plane	297
14.5	Group actions	303
<b>15</b>	<b>Hyperbolic geometry</b>	<b>307</b>
15.1	The hyperbolic plane	307
15.2	The hyperbolic distance	310
15.3	Hyperbolic circles	313
15.4	Hyperbolic trigonometry	315
15.5	Hyperbolic three-dimensional space	317
15.6	Finite Möbius groups	319
	<i>Index</i>	320