

Cambridge University Press
978-0-521-81362-4 - Algebra and Geometry
Alan F. Beardon
Frontmatter
[More information](#)

ALGEBRA AND GEOMETRY

This text gives a basic introduction and a unified approach to algebra and geometry. It covers the ideas of complex numbers, scalar and vector products, determinants, linear algebra, group theory, permutation groups, symmetry groups and various aspects of geometry including groups of isometries, rotations and spherical geometry. The emphasis is always on the interaction between these topics, and each one is constantly illustrated by using it to describe and discuss the others. Many of the ideas are developed gradually throughout the book. For example, the definition of a group is given in Chapter 1 so that it can be used in a discussion of the arithmetic of real and complex numbers; however, many of the properties of groups are given later, and at a time when the importance of the concept has become clear. The text is divided into short sections, with exercises at the end of each one.

Cambridge University Press
978-0-521-81362-4 - Algebra and Geometry
Alan F. Beardon
Frontmatter
[More information](#)

Cambridge University Press
978-0-521-81362-4 - Algebra and Geometry
Alan F. Beardon
Frontmatter
[More information](#)

ALGEBRA AND GEOMETRY

ALAN F. BEARDON



CAMBRIDGE
UNIVERSITY PRESS

Cambridge University Press
978-0-521-81362-4 - Algebra and Geometry
Alan F. Beardon
Frontmatter
[More information](#)

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning and research at the highest international levels of excellence.

www.cambridge.org

Information on this title: www.cambridge.org/9780521813624

© Cambridge University Press 2005

This publication is in copyright. Subject to statutory exception and to the provisions of relevant collective licensing agreements, no reproduction of any part may take place without the written permission of Cambridge University Press.

First published 2005

8th printing 2015

Printed in the United Kingdom by Clays, St Ives plc

A catalogue record for this publication is available from the British Library

Library of Congress Cataloguing in Publication data

ISBN 978-0-521-81362-4 hardback

ISBN 978-0-521-89049-6 paperback

Cambridge University Press has no responsibility for the persistence or accuracy of URLs for external or third-party internet websites referred to in this publication, and does not guarantee that any content on such websites is, or will remain, accurate or appropriate. Information regarding prices, travel timetables and other factual information given in this work are correct at the time of first printing but Cambridge University Press does not guarantee the accuracy of such information thereafter.

Cambridge University Press
978-0-521-81362-4 - Algebra and Geometry
Alan F. Beardon
Frontmatter
[More information](#)

To Dylan, Harry, Fionn and Fenella

Cambridge University Press
978-0-521-81362-4 - Algebra and Geometry
Alan F. Beardon
Frontmatter
[More information](#)

Contents

<i>Preface</i>	<i>page xi</i>
1 Groups and permutations	1
1.1 Introduction	1
1.2 Groups	2
1.3 Permutations of a finite set	6
1.4 The sign of a permutation	11
1.5 Permutations of an arbitrary set	15
2 The real numbers	22
2.1 The integers	22
2.2 The real numbers	26
2.3 Fields	27
2.4 Modular arithmetic	28
3 The complex plane	31
3.1 Complex numbers	31
3.2 Polar coordinates	36
3.3 Lines and circles	40
3.4 Isometries of the plane	41
3.5 Roots of unity	44
3.6 Cubic and quartic equations	46
3.7 The Fundamental Theorem of Algebra	48
4 Vectors in three-dimensional space	52
4.1 Vectors	52
4.2 The scalar product	55
4.3 The vector product	57
4.4 The scalar triple product	60

viii	<i>Contents</i>	
4.5	The vector triple product	62
4.6	Orientation and determinants	63
4.7	Applications to geometry	68
4.8	Vector equations	72
5	Spherical geometry	74
5.1	Spherical distance	74
5.2	Spherical trigonometry	75
5.3	Area on the sphere	77
5.4	Euler's formula	79
5.5	Regular polyhedra	83
5.6	General polyhedra	85
6	Quaternions and isometries	89
6.1	Isometries of Euclidean space	89
6.2	Quaternions	95
6.3	Reflections and rotations	99
7	Vector spaces	102
7.1	Vector spaces	102
7.2	Dimension	106
7.3	Subspaces	111
7.4	The direct sum of two subspaces	115
7.5	Linear difference equations	118
7.6	The vector space of polynomials	120
7.7	Linear transformations	124
7.8	The kernel of a linear transformation	127
7.9	Isomorphisms	130
7.10	The space of linear maps	132
8	Linear equations	135
8.1	Hyperplanes	135
8.2	Homogeneous linear equations	136
8.3	Row rank and column rank	139
8.4	Inhomogeneous linear equations	141
8.5	Determinants and linear equations	143
8.6	Determinants	144
9	Matrices	149
9.1	The vector space of matrices	149
9.2	A matrix as a linear transformation	154
9.3	The matrix of a linear transformation	158

Contents

ix

9.4	Inverse maps and matrices	163
9.5	Change of bases	167
9.6	The resultant of two polynomials	170
9.7	The number of surjections	173
10	Eigenvectors	175
10.1	Eigenvalues and eigenvectors	175
10.2	Eigenvalues and matrices	180
10.3	Diagonalizable matrices	184
10.4	The Cayley–Hamilton theorem	189
10.5	Invariant planes	193
11	Linear maps of Euclidean space	197
11.1	Distance in Euclidean space	197
11.2	Orthogonal maps	198
11.3	Isometries of Euclidean n -space	204
11.4	Symmetric matrices	206
11.5	The field axioms	211
11.6	Vector products in higher dimensions	212
12	Groups	215
12.1	Groups	215
12.2	Subgroups and cosets	218
12.3	Lagrange’s theorem	223
12.4	Isomorphisms	225
12.5	Cyclic groups	230
12.6	Applications to arithmetic	232
12.7	Product groups	235
12.8	Dihedral groups	237
12.9	Groups of small order	240
12.10	Conjugation	242
12.11	Homomorphisms	246
12.12	Quotient groups	249
13	Möbius transformations	254
13.1	Möbius transformations	254
13.2	Fixed points and uniqueness	259
13.3	Circles and lines	261
13.4	Cross-ratios	265
13.5	Möbius maps and permutations	268
13.6	Complex lines	271
13.7	Fixed points and eigenvectors	273

13.8	A geometric view of infinity	276
13.9	Rotations of the sphere	279
14	Group actions	284
14.1	Groups of permutations	284
14.2	Symmetries of a regular polyhedron	290
14.3	Finite rotation groups in space	295
14.4	Groups of isometries of the plane	297
14.5	Group actions	303
15	Hyperbolic geometry	307
15.1	The hyperbolic plane	307
15.2	The hyperbolic distance	310
15.3	Hyperbolic circles	313
15.4	Hyperbolic trigonometry	315
15.5	Hyperbolic three-dimensional space	317
15.6	Finite Möbius groups	319
	<i>Index</i>	320

Preface

Nothing can permanently please, which does not contain in itself the reason why it is so, and not otherwise

S.T. Coleridge, 1772–1834

The idea for this text came after I had given a lecture to undergraduates on the symmetry groups of regular solids. It is a beautiful subject, so why was I unhappy with the outcome? I had covered the subject in a more or less standard way, but as I came away I became aware that I had assumed Euler's theorem on polyhedra, I had assumed that every symmetry of a polyhedron extended to an isometry of space, and that such an isometry was necessarily a rotation or a reflection (again due to Euler), and finally, I had not given any convincing reason why such polyhedra did actually exist. Surely these ideas are at least as important (or perhaps more so) than the mere identification of the symmetry groups of the polyhedra?

The primary aim of this text is to present many of the ideas and results that are typically given in a university course in mathematics in a way that emphasizes the coherence and mutual interaction within the subject as a whole. We believe that by taking this approach, students will be able to support the parts of the subject that they find most difficult with ideas that they can grasp, and that the unity of the subject will lead to a better understanding of mathematics as a whole. Inevitably, this approach will not take the reader as far down any particular road as a single course in, say, group theory might, but we believe that this is the right approach for a student who is beginning a university course in mathematics. Increasingly, students will be taking more and more courses outside mathematics, and the pressure to include a wide spread of mathematics within a limited time scale will increase. We believe that the route advocated above will, in addition to being educationally desirable, help solve this problem.

To illustrate our approach, consider once again the symmetries of the five (regular) Platonic solids. These symmetries may be viewed as examples of permutations (acting on the vertices, or the faces, or even on the diagonals) of the solid, but they can also be viewed as finite groups of rotations of Euclidean 3-space. This latter point of view suggests that the discussion should lead into, or away from, a discussion of the nature of isometries of 3-space, for this is fundamental to the very definition of the symmetry groups. From a different point of view, probably the easiest way to identify the Platonic solids is by means of Euler's formula for the sphere. Now Euler's formula can be (and here is) proved by means of spherical geometry and trigonometry, and the requisite formulae here are simple (and important) applications of the standard scalar and vector product of the 'usual' vectors in 3-space (as studied in applied mathematics). Next, by studying rotation groups acting on the unit sphere in 3-space one can prove that the symmetry groups of the regular solids are the only finite groups of rotations of 3-space, a fact that is not immediately apparent from the geometry. Finally, by using stereographic projection (as appears in any complex analysis course that acknowledges the point at infinity) the symmetry groups of the regular solids appear as the only finite groups of Möbius transformations acting in hyperbolic space. Moreover in this guise one can also introduce rotations of 3-space in terms of quaternions which then appear as 2-by-2 complex matrices.

The author firmly believes that this is the way mathematics should be introduced, and moreover that it can be so introduced at a reasonably elementary level. In many cases, students find mathematics difficult because they fail to grasp the initial concepts properly, and in this approach preference is given to understanding and reinforcing these basic concepts from a variety of different points of view rather than moving on in the traditional way to provide yet more theorems that the student has to try to cope with from a sometimes uncertain base.

This text includes the basic definitions, and some early results, on, for example, groups, vector spaces, quaternions, eigenvectors, the diagonalization of matrices, orthogonal groups, isometries of the complex plane and of Euclidean space, scalar and vector products in 3-space, Euclidean, spherical and (briefly) hyperbolic geometries, complex numbers and Möbius transformations. Above all, it is these basic concepts and their mutual interaction which is the main theme of this text.

Finally an earlier version of this book can be freely downloaded as an html file from <http://www.cambridge.org/0521890497>. This file is under development and the aim is to create a fully linked electronic textbook.