Bäcklund and Darboux Transformations
Geometry and Modern Applications in Soliton Theory

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Contents

1

Pseudospherical Surfaces and the Classical Bäcklund Transformation. The Bianchi System

The explicit study of surfaces of constant negative total curvature goes back to the work of Minding [261] in 1838. Thus, in that year, Minding's theorem established the important result that these surfaces are isometric, that is, points on two such surfaces can be placed in one-to-one correspondence in a way that the metric is preserved. Beltrami [28] subsequently gave the term pseudospherical to these surfaces and made important connections with Lobachevski's non-Euclidean geometry.

It was Bour [54], in 1862, who seems to have first set down what is now termed the sine-Gordon equation arising out of the compatibility conditions for the Gauss equations for pseudospherical surfaces expressed in asymptotic coordinates.

In 1879, Bianchi [31] in his habilitation thesis presented, in mathematical terms, a geometric construction for pseudospherical surfaces. This result was extended by Bäcklund [21] in 1883 to incorporate a key parameter which allows the iterative construction of such pseudospherical surfaces. The Bäcklund transformation was subsequently shown by Bianchi [32], in 1885, to be associated with an elegant invariance of the sine-Gordon equation. This invariance has become known as the Bäcklund transformation for the sine-Gordon equation. It includes an earlier parameter-independent result of Darboux [94]. The Bäcklund transformation has important applications in soliton theory. Indeed, it appears that the property of invariance under Bäcklund and associated Darboux transformations as originated in [92] is enjoyed by all soliton equations. The contribution of Bianchi and Darboux to the geometry of surfaces and, in particular, the role of Bäcklund transformations preserving certain geometric properties have been discussed by Chern [77] and Sym et al. in [385]. It is with Bäcklund and Darboux transformations, their geometric origins and their application in modern soliton theory that we shall be concerned in the present monograph.

1.1 The Gauss-Weingarten Equations for Hyperbolic Surfaces. Pseudospherical Surfaces. The Sine-Gordon Equation

Here, the study of pseudospherical surfaces is set in the broader context of hyperbolic surfaces via a nonlinear system due to Bianchi [37]. The background is that of basic classical differential geometry of curves and surfaces to be found in such standard works as do Carmo [108] or Struick [352]. The latter work is a rich source of material on the history of the subject.

Let $\boldsymbol{r} = \boldsymbol{r}(u, v)$ denote the position vector of a generic point P on a surface Σ in $\mathbb{R}^3$. Then, the vectors $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$ are tangential to Σ at P and, at such points at which they are linearly independent,

$$\boldsymbol{N} = \frac{\boldsymbol{r}_u \times \boldsymbol{r}_v}{|\boldsymbol{r}_u \times \boldsymbol{r}_v|} \tag{1.1}$$

determines the unit normal to Σ. The 1st and 2nd fundamental forms of Σ are defined by

$$\begin{aligned} \mathrm{I} &= \quad d\boldsymbol{r} \cdot d\boldsymbol{r} \quad = E\,du^2 + 2F\,du dv + G\,dv^2, \\ \mathrm{II} &= -d\boldsymbol{r} \cdot d\boldsymbol{N} = e\,du^2 + 2f\,du dv + g\,dv^2, \end{aligned} \tag{1.2}$$

where

$$\begin{gathered} E = \boldsymbol{r}_u \cdot \boldsymbol{r}_u, \quad F = \boldsymbol{r}_u \cdot \boldsymbol{r}_v, \quad G = \boldsymbol{r}_v \cdot \boldsymbol{r}_v, \\ e = -\boldsymbol{r}_u \cdot \boldsymbol{N}_u = \boldsymbol{r}_{uu} \cdot \boldsymbol{N}, \quad g = -\boldsymbol{r}_v \cdot \boldsymbol{N}_v = \boldsymbol{r}_{vv} \cdot \boldsymbol{N}. \\ f = -\boldsymbol{r}_u \cdot \boldsymbol{N}_v = -\boldsymbol{r}_v \cdot \boldsymbol{N}_u = \boldsymbol{r}_{uv} \cdot \boldsymbol{N}. \end{gathered} \tag{1.3}$$

An important classical result due to Bonnet [53] states that the sextuplet $\{E, F, G;\ e, f, g\}$ determines the surface Σ up to its position in space.

The Gauss equations associated with Σ are [352]

$$\begin{aligned} \boldsymbol{r}_{uu} &= \Gamma^1_{11}\boldsymbol{r}_u + \Gamma^2_{11}\boldsymbol{r}_v + e\boldsymbol{N}, \\ \boldsymbol{r}_{uv} &= \Gamma^1_{12}\boldsymbol{r}_u + \Gamma^2_{12}\boldsymbol{r}_v + f\boldsymbol{N}, \\ \boldsymbol{r}_{vv} &= \Gamma^1_{22}\boldsymbol{r}_u + \Gamma^2_{22}\boldsymbol{r}_v + g\boldsymbol{N}, \end{aligned} \tag{1.4}$$

while the Weingarten equations comprise

$$\begin{aligned} \boldsymbol{N}_u &= \frac{fF - eG}{H^2}\boldsymbol{r}_u + \frac{eF - fE}{H^2}\boldsymbol{r}_v, \\ \boldsymbol{N}_v &= \frac{gF - fG}{H^2}\boldsymbol{r}_u + \frac{fF - gE}{H^2}\boldsymbol{r}_v, \end{aligned} \tag{1.5}$$

where

$$H^2 = |\boldsymbol{r}_u \times \boldsymbol{r}_v|^2 = EG - F^2. \tag{1.6}$$

The Γ^i_{jk} in (1.4) are the usual Christoffel symbols given by the relations

$$\Gamma^i_{jk} = \frac{g^{il}}{2}(g_{jl,k} + g_{kl,j} - g_{jk,l}), \tag{1.7}$$

where, with $x^1 = u$, $x^2 = v$,

$$\mathrm{I} = g_{jk}dx^j dx^k, \tag{1.8}$$

and

$$g^{jk}g_{kl} = \delta^j_l. \tag{1.9}$$

In the above, the Einstein convention of summation over repeated indices has been adopted.

The compatibility conditions $(\boldsymbol{r}_{uu})_v = (\boldsymbol{r}_{uv})_u$ and $(\boldsymbol{r}_{uv})_v = (\boldsymbol{r}_{vv})_u$ applied to the *linear* Gauss system (1.4) produce the *nonlinear* Mainardi-Codazzi system

$$\begin{aligned}
&\left(\frac{e}{H}\right)_v - \left(\frac{f}{H}\right)_u + \frac{e}{H}\Gamma^2_{22} - 2\frac{f}{H}\Gamma^2_{12} + \frac{g}{H}\Gamma^2_{11} = 0,\\
&\left(\frac{g}{H}\right)_u - \left(\frac{f}{H}\right)_v + \frac{e}{H}\Gamma^1_{22} - 2\frac{f}{H}\Gamma^1_{12} + \frac{g}{H}\Gamma^1_{11} = 0
\end{aligned} \tag{1.10}$$

or, equivalently,

$$\begin{aligned}
e_v - f_u &= e\Gamma^1_{12} + f\left(\Gamma^2_{12} - \Gamma^1_{11}\right) - g\Gamma^2_{11},\\
f_v - g_u &= e\Gamma^1_{22} + f\left(\Gamma^2_{22} - \Gamma^1_{12}\right) - g\Gamma^2_{12},
\end{aligned} \tag{1.11}$$

augmented by the ‘Theorema egregium’ of Gauss. The latter provides an expression for the *Gaussian (total) curvature*

$$\mathcal{K} = \frac{eg - f^2}{EG - F^2} \tag{1.12}$$

in terms of E, F, G alone according to, in Liouville’s representation,

$$\mathcal{K} = \frac{1}{H}\left[\left(\frac{H}{E}\Gamma^2_{11}\right)_v - \left(\frac{H}{E}\Gamma^2_{12}\right)_u\right]. \tag{1.13}$$

In physical terms, the ‘Theorema egregium’ implies that the total curvature of a surface Σ is invariant under bending without stretching.

If the total curvature of Σ is negative, that is, if Σ is a hyperbolic surface, then the *asymptotic lines* on Σ may be taken as parametric curves. Then $e = g = 0$ and the Mainardi-Codazzi equations (1.10) reduce to,

$$\left(\frac{f}{H}\right)_u + 2\Gamma^2_{12}\frac{f}{H} = 0, \quad \left(\frac{f}{H}\right)_v + 2\Gamma^1_{12}\frac{f}{H} = 0 \tag{1.14}$$

while

$$\mathcal{K} = -\frac{f^2}{H^2} =: -\frac{1}{\rho^2} \tag{1.15}$$

and

$$\Gamma^1_{12} = \frac{GE_v - FG_u}{2H^2}, \tag{1.16}$$

$$\Gamma^2_{12} = \frac{EG_u - FE_v}{2H^2}. \tag{1.17}$$

The angle ω between the parametric lines is such that

$$\cos\omega = \frac{F}{\sqrt{EG}}, \quad \sin\omega = \frac{H}{\sqrt{EG}} \tag{1.18}$$

and since $E, G > 0$, we may take, without loss of generality,

$$E = \rho^2 a^2, \quad G = \rho^2 b^2, \tag{1.19}$$

whence I and II reduce to

$$\begin{aligned} \mathrm{I} &= \rho^2(a^2 du^2 + 2ab\cos\omega\, du dv + b^2 dv^2), \\ \mathrm{II} &= 2\rho ab \sin\omega\, du dv. \end{aligned} \tag{1.20}$$

The Mainardi-Codazzi equations (1.11) now show that

$$a_v + \frac{1}{2}\frac{\rho_v}{\rho}a - \frac{1}{2}\frac{\rho_u}{\rho}b\cos\omega = 0, \tag{1.21}$$

$$b_u + \frac{1}{2}\frac{\rho_u}{\rho}b - \frac{1}{2}\frac{\rho_v}{\rho}a\cos\omega = 0, \tag{1.22}$$

while the representation (1.13) for the total curvature yields

$$\omega_{uv} + \frac{1}{2}\left(\frac{\rho_u}{\rho}\frac{b}{a}\sin\omega\right)_u + \frac{1}{2}\left(\frac{\rho_v}{\rho}\frac{a}{b}\sin\omega\right)_v - ab\sin\omega = 0. \tag{1.23}$$

The nonlinear system of Gauss-Mainardi-Codazzi equations (1.21)–(1.23) was originally set down by Bianchi (see [37]). Its importance in soliton theory has been noted by Cenkl [74] and subsequently by Levi and Sym [234]. It will be returned to later in that connection subject to an additional constraint, namely $\rho_{uv} = 0$. The system then becomes solitonic.

In the particular case when $\mathcal{K} = -1/\rho^2 < 0$ is a constant, Σ is termed a *pseudospherical* surface. The Mainardi-Codazzi equations (1.21), (1.22) then yield $a = a(u)$, $b = b(v)$. If Σ is now parametrised by arc length along asymptotic lines (corresponding to the transformation $du \to du' = \sqrt{E(u)}\,du, dv \to dv' = \sqrt{G(v)}\,dv$), then the fundamental forms become, on dropping the primes,

$$\begin{aligned} \mathrm{I} &= du^2 + 2\cos\omega\, du dv + dv^2, \\ \mathrm{II} &= \frac{2}{\rho}\sin\omega\, du dv, \end{aligned} \tag{1.24}$$

while (1.23) reduces to the celebrated *sine-Gordon* equation

$$\boxed{\omega_{uv} = \frac{1}{\rho^2}\sin\omega.} \tag{1.25}$$

The associated Gauss equations yield

$$\begin{aligned} \boldsymbol{r}_{uu} &= \omega_u \cot\omega\, \boldsymbol{r}_u - \omega_u \operatorname{cosec}\omega\, \boldsymbol{r}_v, \\ \boldsymbol{r}_{uv} &= \frac{1}{\rho}\sin\omega \boldsymbol{N}, \\ \boldsymbol{r}_{vv} &= -\omega_v \operatorname{cosec}\omega\, \boldsymbol{r}_u + \omega_v \cot\omega\, \boldsymbol{r}_v, \end{aligned} \tag{1.26}$$

while those of Weingarten give

$$\begin{aligned} \boldsymbol{N}_u &= \frac{1}{\rho}\cot\omega\, \boldsymbol{r}_u - \frac{1}{\rho}\operatorname{cosec}\omega\, \boldsymbol{r}_v, \\ \boldsymbol{N}_v &= -\frac{1}{\rho}\operatorname{cosec}\omega\, \boldsymbol{r}_u + \frac{1}{\rho}\cot\omega\, \boldsymbol{r}_v. \end{aligned} \tag{1.27}$$

In the twentieth century, the sine-Gordon equation has been shown, remarkably, to arise in a diversity of areas of physical interest (see [311]). It was the work of Seeger et al. [201, 345, 346] that first demonstrated how the classical Bäcklund transformation for this equation has important application in the theory of crystal dislocations. Indeed, in [345], within the context of Frenkel's and Kontorova's dislocation theory, the superposition of so-called 'eigenmotions' was obtained by means of the classical Bäcklund transformation. The interaction of what today are called breathers with kink-type dislocations was both described analytically and displayed graphically. The typical solitonic features to be subsequently discovered by Zabusky and Kruskal [389] in 1965 for the Korteweg-de Vries equation, namely preservation of velocity and shape following interaction, as well as the concomitant phase shift, were all recorded for the sine-Gordon equation in this remarkable paper of 1953.[1] Connections between the geometry of pseudospherical surfaces and other solitonic equations have been later investigated in [26, 78, 79, 141, 190, 292, 294, 321, 363].

Lamb [223] and Barnard [23] showed that the nonlinear superposition principle associated with the Bäcklund transformation for the sine-Gordon equation has application in the theory of ultrashort optical pulse propagation. In particular, solitonic decomposition phenomena observed experimentally in *Rb* vapour by Gibbs and Slusher [150] were thereby reproduced theoretically. In addition, the classical Bäcklund transformation has also found application in the theory of long Josephson junctions [344].

The preceding provides an historical motivation, both with regard to theory and application, for beginning our study of Bäcklund transformations with the classical result for the sine-Gordon equation. It will be seen that this Bäcklund transformation, in fact, corresponds to a conjugation of invariant transformations due to Bianchi and Lie. The Lie symmetry serves to intrude a key *Bäcklund parameter* into the Bianchi transformation which enables its iteration and the generation thereby of what are, in physical terms, multi-soliton solutions. Therein, the Bäcklund parameters have an important physical interpretation.

1.2 The Classical Bäcklund Transformation for the Sine-Gordon Equation

Underlying the original Bäcklund transformation for the sine-Gordon equation is a simple geometric construction for pseudospherical surfaces. Thus, if a point P is taken on an initial pseudospherical surface Σ and a line segment PP' of constant length and tangential to Σ at P is constructed in a manner dictated by a Bäcklund transformation as described below, then the locus of the points P' as P traces out Σ is another pseudospherical surface Σ' with the same total curvature as Σ. The procedure may be iterated to generate a sequence of pseudospherical surfaces with the same total curvature as the original seed surface Σ.

[1] "Man sieht ... daß beim Durchdringen von Wellengruppe und Versetzung weder die Energie noch die Geschwindigkeit beider geändert wird. Es tritt lediglich eine Verschiebung des Versetzungsmittelpunktes ... und des Schwerpunktes der Wellengruppe ... auf" [345, p 189].

Let Σ be a pseudospherical surface with total curvature $\mathcal{K} = -1/\rho^2$ and with generic position vector $\boldsymbol{r} = \boldsymbol{r}(u, v)$, where u, v correspond to the parametrisation by arc length along asymptotic lines. In this parametrisation, $\boldsymbol{r}_u, \boldsymbol{r}_v$ and $\boldsymbol{N}$ are all unit vectors, but $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$ are not orthogonal. Accordingly, it proves convenient to introduce an orthonormal triad $\{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}\}$, where

$$\begin{aligned} \boldsymbol{A} = \boldsymbol{r}_u, \quad \boldsymbol{B} &= -\boldsymbol{r}_u \times \boldsymbol{N} = -\boldsymbol{r}_u \times \frac{(\boldsymbol{r}_u \times \boldsymbol{r}_v)}{\sin\omega}, \quad \boldsymbol{C} = \boldsymbol{N} \\ &= \operatorname{cosec}\omega\, \boldsymbol{r}_v - \cot\omega\, \boldsymbol{r}_u . \end{aligned} \tag{1.28}$$

The Gauss-Weingarten equations (1.26), (1.27) can now be used to obtain expressions for the derivatives of $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ with respect to u and v, namely

$$\begin{aligned} \begin{pmatrix} \boldsymbol{A} \\ \boldsymbol{B} \\ \boldsymbol{C} \end{pmatrix}_u &= \begin{pmatrix} 0 & -\omega_u & 0 \\ \omega_u & 0 & 1/\rho \\ 0 & -1/\rho & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{A} \\ \boldsymbol{B} \\ \boldsymbol{C} \end{pmatrix}, \\ \begin{pmatrix} \boldsymbol{A} \\ \boldsymbol{B} \\ \boldsymbol{C} \end{pmatrix}_v &= \begin{pmatrix} 0 & 0 & (1/\rho)\sin\omega \\ 0 & 0 & -(1/\rho)\cos\omega \\ -(1/\rho)\sin\omega & (1/\rho)\cos\omega & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{A} \\ \boldsymbol{B} \\ \boldsymbol{C} \end{pmatrix}. \end{aligned} \tag{1.29}$$

This linear system is compatible if and only if ω satisfies the sine-Gordon equation (1.25).

A new pseudospherical surface Σ' with position vector $\boldsymbol{r}'$ is now sought in the form

$$\boldsymbol{r}' = \boldsymbol{r} + L\cos\phi\,\boldsymbol{A} + L\sin\phi\,\boldsymbol{B}, \tag{1.30}$$

where $L = |\boldsymbol{r}' - \boldsymbol{r}|$ is constant. Here, $\phi(u, v)$ is to be constrained by the requirement that on Σ', as on Σ, the coordinates u, v correspond to parametrisation along asymptotic lines. A necessary condition for this to be the case is that Σ' have a 1st fundamental form of the type $(1.24)_1$. In particular, this requires that

$$\boldsymbol{r}'_u \cdot \boldsymbol{r}'_u = 1, \;\; \boldsymbol{r}'_v \cdot \boldsymbol{r}'_v = 1, \tag{1.31}$$

where, on use of (1.30) and the relations (1.29), we have

$$\begin{aligned} \boldsymbol{r}'_u &= [1 - L(\phi_u - \omega_u)\sin\phi]\boldsymbol{A} + L(\phi_u - \omega_u)\cos\phi\,\boldsymbol{B} + \frac{L}{\rho}\sin\phi\,\boldsymbol{C}, \\ \boldsymbol{r}'_v &= (\cos\omega - L\phi_v\sin\phi)\boldsymbol{A} + (\sin\omega + L\phi_v\cos\phi)\boldsymbol{B} + \frac{L}{\rho}\sin(\omega - \phi)\boldsymbol{C}. \end{aligned} \tag{1.32}$$

The conditions (1.31) now yield, in turn,

$$\phi_u = \omega_u + \frac{1}{L}\left(1 \pm \sqrt{1 - \frac{L^2}{\rho^2}}\right)\sin\phi \tag{1.33}$$

and

$$\phi_v = \frac{1}{L}\left(1 \mp \sqrt{1 - \frac{L^2}{\rho^2}}\right)\sin(\phi - \omega). \tag{1.34}$$

Accordingly, if we set

$$\beta = \frac{\rho}{L}\left(1 \pm \sqrt{1 - \frac{L^2}{\rho^2}}\right) = \frac{L}{\rho}\left(1 \mp \sqrt{1 - \frac{L^2}{\rho^2}}\right)^{-1}, \tag{1.35}$$

then the relations (1.33), (1.34), deliver the necessary requirements

$$\phi_u = \omega_u + \frac{\beta}{\rho}\sin\phi, \tag{1.36}$$

$$\phi_v = \frac{1}{\beta\rho}\sin(\phi - \omega) \tag{1.37}$$

on the angle ϕ in order that Σ' be a pseudospherical surface parametrised by arc length along asymptotic lines. In fact, the pair of equations, (1.36), (1.37), is sufficient in this regard. Moreover, these equations are compatible modulo the sine-Gordon equation (1.25).

On use of (1.36), (1.37), the expressions (1.32) become

$$\boldsymbol{r}'_u = \left(1 - \frac{L}{\rho}\beta\sin^2\phi\right)\boldsymbol{A} + \frac{L}{\rho}\beta\sin\phi\cos\phi\,\boldsymbol{B} + \frac{L}{\rho}\sin\phi\,\boldsymbol{C}, \tag{1.38}$$

$$\begin{aligned}\boldsymbol{r}'_v = &\left[\cos\omega - \frac{L}{\rho\beta}\sin\phi\sin(\phi - \omega)\right]\boldsymbol{A}\\ &+ \left[\sin\omega + \frac{L}{\rho\beta}\cos\phi\sin(\phi - \omega)\right]\boldsymbol{B} - \frac{L}{\rho}\sin(\phi - \omega)\,\boldsymbol{C},\end{aligned} \tag{1.39}$$

so that $\boldsymbol{r}'_u \cdot \boldsymbol{r}'_v = \cos(2\phi - \omega)$ and the 1st fundamental form of Σ' becomes

$$\mathrm{I}' = du^2 + 2\cos(2\phi - \omega)\,du\,dv + dv^2. \tag{1.40}$$

Furthermore, the unit normal $\boldsymbol{N}'$ to Σ' is given by

$$\boldsymbol{N}' = \frac{\boldsymbol{r}'_u \times \boldsymbol{r}'_v}{|\boldsymbol{r}'_u \times \boldsymbol{r}'_v|} = -\frac{L}{\rho} \sin\phi \boldsymbol{A} + \frac{L}{\rho} \cos\phi \boldsymbol{B} + \left(1 - \frac{L\beta}{\rho}\right) \boldsymbol{C}, \qquad (1.41)$$

whence, on use of (1.30), it is seen that $(\boldsymbol{r}' - \boldsymbol{r}) \cdot \boldsymbol{N}' = 0$. Accordingly, the vector $\boldsymbol{r}' - \boldsymbol{r}$ joining corresponding points on Σ and Σ' is tangential to Σ'. It is recalled that it is tangential to Σ by construction. Moreover,

$$\boldsymbol{N}'_u = -\frac{L\beta}{\rho^2} \sin\phi \cos\phi \boldsymbol{A} + \left(\frac{L\beta}{\rho^2} \cos^2\phi - \frac{1}{\rho}\right) \boldsymbol{B} + \frac{L}{\rho^2} \cos\phi \boldsymbol{C}, \qquad (1.42)$$

$$\begin{aligned} \boldsymbol{N}'_v = & \left[\frac{L}{2\rho^2\beta} \sin(\omega - 2\phi) + \frac{1}{\rho}\left(1 - \frac{L}{2\rho\beta}\right) \sin\omega\right] \boldsymbol{A} \\ & + \left[\frac{L}{2\rho^2\beta} \cos(\omega - 2\phi) - \frac{1}{\rho}\left(1 - \frac{L}{2\rho\beta}\right) \cos\omega\right] \boldsymbol{B} \\ & - \frac{L}{\rho^2} \cos(\omega - \phi) \boldsymbol{C}, \end{aligned} \qquad (1.43)$$

whence

$$\boldsymbol{r}'_u \cdot \boldsymbol{N}'_u = 0, \quad \boldsymbol{r}'_u \cdot \boldsymbol{N}'_v = \boldsymbol{r}'_v \cdot \boldsymbol{N}'_u = -\frac{1}{\rho} \sin(2\phi - \omega), \quad \boldsymbol{r}'_v \cdot \boldsymbol{N}'_v = 0.$$

The 2$^{\text{nd}}$ fundamental form for Σ' is

$$\text{II}' = \frac{2}{\rho} \sin(2\phi - \omega)\, du dv \qquad (1.44)$$

and this together with I' as given by (1.40) shows that Σ' is a pseudospherical surface parametrised by arc length along asymptotic lines. The angle between the asymptotic lines on Σ' is given by

$$\omega' = 2\phi - \omega, \qquad (1.45)$$

where ω' plays the same role in relation to Σ' as is played by ω in relation to Σ. In particular, ω' must satisfy the sine-Gordon equation

$$\omega'_{uv} = \frac{1}{\rho^2} \sin\omega'. \qquad (1.46)$$

Use of the relation (1.45) to eliminate ϕ in (1.36) and (1.37) now yields

$$\boxed{\begin{aligned}\left(\frac{\omega'-\omega}{2}\right)_u &= \frac{\beta}{\rho}\sin\left(\frac{\omega'+\omega}{2}\right)\\ \left(\frac{\omega'+\omega}{2}\right)_v &= \frac{1}{\beta\rho}\sin\left(\frac{\omega'-\omega}{2}\right).\end{aligned}} \quad \mathbb{B}_\beta \tag{1.47}$$

This is the standard form of the Bäcklund transformation which links the sine-Gordon equations (1.25) and (1.46).

It is noted that, under $\mathbb{B}_\beta$,

$$\boldsymbol{N}'\cdot\boldsymbol{N} = 1-\frac{L\beta}{\rho} = \text{const}, \tag{1.48}$$

that is, the tangent planes at corresponding points on Σ and Σ' meet at a constant angle ζ where $\beta = \tan(\zeta/2)$. In Bianchi's original geometric construction, of which the Bäcklund result is an extension,

$$L = \rho, \quad \beta = 1 \tag{1.49}$$

so that these tangent planes are orthogonal. Bäcklund's relaxation of the orthogonality requirement allows the key parameter β to be inserted into the Bianchi transformation. In fact, the Bäcklund transformation $\mathbb{B}_\beta$ may be viewed as a composition of a Bianchi transformation with a simple Lie group invariance. Thus, the sine-Gordon equation (1.25) is invariant under the scaling

$$u^* = \beta u, \quad v^* = \frac{v}{\beta}, \qquad \beta \neq 0 \tag{1.50}$$

so that, any solution $\omega = \omega(u, v)$ generates a one-parameter class of solutions $\omega^*(u^*, v^*) = \omega(\beta u, v/\beta)$.[2] Lie observed that conjugation of the invariance (1.50) with the original Bianchi transformation

$$\begin{aligned}\left(\frac{\omega'-\omega}{2}\right)_{u^*} &= \frac{1}{\rho}\sin\left(\frac{\omega'+\omega}{2}\right),\\ \left(\frac{\omega'+\omega}{2}\right)_{v^*} &= \frac{1}{\rho}\sin\left(\frac{\omega'-\omega}{2}\right)\end{aligned} \tag{1.51}$$

produces the Bäcklund transformation (1.47).

[2] Importantly, this Lie point invariance also inserts the Bäcklund parameter β into the 'linear representation' (1.29) and delivers a one-parameter family of pseudospherical surfaces associated with a given solution ω of the sine-Gordon equation.

In terms of the construction of pseudospherical surfaces, the Bäcklund transformation corresponds to the following result: let $\boldsymbol{r}$ be the coordinate vector of the pseudospherical surface Σ corresponding to a solution ω of the sine-Gordon equation (1.25). Let ω' denote the Bäcklund transform of ω via $\mathbb{B}_\beta$. Then, the coordinate vector $\boldsymbol{r}'$ of the pseudospherical surface Σ' corresponding to ω' is given by

$$\boldsymbol{r}' = \boldsymbol{r} + \frac{L}{\sin\omega}\left[\sin\left(\frac{\omega-\omega'}{2}\right)\boldsymbol{r}_u + \sin\left(\frac{\omega+\omega'}{2}\right)\boldsymbol{r}_v\right], \qquad (1.52)$$

where $L = \rho \sin\zeta$.

1.2.0.1 Key Observations

- The *nonlinear* sine-Gordon equation (1.25) is derived as the compatibility condition for the *linear* Gauss equations (1.26).
- The Bäcklund transformation $\mathbb{B}_\beta$ given by (1.47) acts on the sine-Gordon equation (1.25) and leaves it invariant. Indeed, the action of $\mathbb{B}_\beta$ is restricted to (1.25) in that (1.47) is a valid system for ω' if and only if (1.25) holds: otherwise the compatibility condition $\omega'_{uv} = \omega'_{vu}$ is not satisfied.
- $\mathbb{B}_\beta$ contains a parameter $\beta = \tan(\zeta/2)$ injected into the underlying Bianchi transformation by a Lie group invariance.
- At the *linear* level, the Bäcklund transformation is represented by (1.52) and acts on the Gauss system (1.26) associated with pseudospherical surfaces parametrised by arc length along asymptotic lines. The transformation (1.52) acting on the underlying linear representation (1.26) induces the Bäcklund transformation $\mathbb{B}_\beta$ operating at the *nonlinear* level.

In that $\mathbb{B}_\beta$ represents a correspondence between solutions of the same equation, it is commonly termed an *auto-Bäcklund* transformation.

In the next section, a nonlinear superposition principle associated with the auto-Bäcklund transformation $\mathbb{B}_\beta$ will be derived whereby, in particular, multi-soliton solutions of the nonlinear sine-Gordon equation (1.25) may be generated by *purely algebraic procedures*. The algorithmic nature of the latter makes them well-suited to implementation by symbolic computation packages. Such nonlinear superposition principles are generically associated with the auto-Bäcklund transformations admitted by solitonic equations.

Exercises

1. Establish the relations (1.33), (1.34) governing the angle ϕ.
2. Derive the expression (1.52) descriptive of the action of the Bäcklund transformation $\mathbb{B}_\beta$ at the pseudospherical surface level.

1.3 Bianchi's Permutability Theorem. Generation of Multi-Soliton Solutions

Next, we turn to the application of the auto-Bäcklund transformation (1.47) to construct multi-soliton solutions of the sine-Gordon equation.

Let us start with the seed 'vacuum' solution $\omega = 0$ of (1.25). The Bäcklund transformation (1.47) shows that a second, but nontrivial, solution ω' of (1.46) may be constructed by integration of the pair of first-order equations

$$\begin{aligned} \omega'_u &= \frac{2\beta}{\rho}\sin\left(\frac{\omega'}{2}\right), \\ \omega'_v &= \frac{2}{\beta\rho}\sin\left(\frac{\omega'}{2}\right), \end{aligned} \tag{1.53}$$

leading to the new *single soliton* solution

$$\omega' = 4\tan^{-1}\left[\exp\left(\frac{\beta}{\rho}u + \frac{1}{\beta\rho}v + \alpha\right)\right], \tag{1.54}$$

where α is an arbitrary constant of integration. It should be noted that, here, it is the quantities

$$\begin{aligned} \omega'_u &= \frac{2\beta}{\rho}\operatorname{sech}\left(\frac{\beta}{\rho}u + \frac{1}{\beta\rho}v + \alpha\right), \\ \omega'_v &= \frac{2}{\beta\rho}\operatorname{sech}\left(\frac{\beta}{\rho}u + \frac{1}{\beta\rho}v + \alpha\right), \end{aligned} \tag{1.55}$$

which have the characteristic hump shape associated with a soliton.

Remarkably, analytic expressions for multi-soliton solutions which encapsulate their nonlinear interaction may now be obtained by an entirely algebraic procedure. This is a consequence of an elegant nonlinear superposition principle derived from the auto-Bäcklund transformation $\mathbb{B}_\beta$ and originally set down by Bianchi [35] in 1892. It is described in his monumental work [37] and is now known as:

1.3.1 Bianchi's Permutability Theorem

Suppose ω is a seed solution of the sine-Gordon equation (1.25) and that ω_1 and ω_2 are the Bäcklund transforms of ω via $\mathbb{B}_{\beta_1}$ and $\mathbb{B}_{\beta_2}$, that is, $\omega_1 = \mathbb{B}_{\beta_1}(\omega)$, $\omega_2 = \mathbb{B}_{\beta_2}(\omega)$. Let $\omega_{12} = \mathbb{B}_{\beta_2}(\omega_1)$ and $\omega_{21} = \mathbb{B}_{\beta_1}(\omega_2)$. The situation may be represented schematically by a *Bianchi diagram* as given in Figure 1.1.

It is natural to enquire if there are any circumstances under which the commutative condition $\omega_{12} = \omega_{21}$ applies. To investigate this matter, we set down the