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151 Frobenius manifolds and moduli spaces for singularities

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Preface

Frobenius manifolds are complex manifolds with a rich structure on the holomorphic tangent bundle, a multiplication and a metric which harmonize in the most natural way. They were defined by Dubrovin in 1991, motivated by the work of Witten, Dijkgraaf, E. Verlinde, and H. Verlinde on topological field theory. Originally coming from physics, Frobenius manifolds now turn up in very different areas of mathematics, giving unexpected relations between them, in quantum cohomology, singularity theory, integrable systems, symplectic geometry, and others. The isomorphy of certain Frobenius manifolds in quantum cohomology and in singularity theory is one version of mirror symmetry.

This book is devoted to the relations between Frobenius manifolds and singularity theory. It consists of two parts.

In part 1 F-manifolds are studied, manifolds with a multiplication on the tangent bundle with a natural integrability condition. They were introduced in [**HM**][**Man2**, I§5]. Frobenius manifolds are F-manifolds. Studying F-manifolds, one is led directly to discriminants, a classical subject of singularity theory, and to Lagrange maps and their singularities. Our development of the general structure of F-manifolds is at the same time an introduction to discriminants and Lagrange maps. As an application, we use some work of Givental to prove a conjecture of Dubrovin about Frobenius manifolds and Coxeter groups.

In part 2 we take up the construction of Frobenius manifolds in singularity theory. Already in 1983 K. Saito and M. Saito had found that the base space of a semiuniversal unfolding of an isolated hypersurface singularity can be equipped with the structure of a Frobenius manifold. Their construction involves the Gauß-Manin connection, polarized mixed Hodge structures, K. Saito's higher residue pairings, and his primitive forms. It was hardly accessible for nonspecialists. We give a more elementary detailed account of the construction, explain all ingredients, and develop or cite all necessary results.

We give a number of applications. The deepest one is the construction of global moduli spaces for isolated hypersurface singularities.

The construction of K. Saito and M. Saito is related to a recent construction of Frobenius manifolds via oscillating integrals by Sabbah and Barannikov. We comment upon that.

Background and other books. The reader should know the basic concepts of complex analytic geometry, including coherent sheaves and flatness (cf. for example [**Fi**]). All notions from symplectic geometry which are used can be found in [**AGV1**, chapter 18]. An excellent basic reference on flat connections and vector bundles (and much more) is the forthcoming book [**Sab4**]. It also gives a viewpoint on Frobenius manifolds which complements ours. Two fundamental books on Frobenius manifolds are [**Du3**] and [**Man2**]. Our treatment of singularities and their Gauß-Manin connection is essentially self-contained and gives precisely what is needed, but it is quite compact. Some books which expound several aspects in much more detail are [**AGV1**][**AGV2**][**Ku**][**Lo2**].

Acknowledgements. This book grew out of my habilitation. I would like to thank many people. E. Brieskorn was my teacher in singularity theory and defined in 1970 the wonderful object H_0'', which is now called the Brieskorn lattice. Yu. Manin introduced me to Frobenius manifolds. The common paper [**HM**] was the starting point for part 1. His papers and those of B. Dubrovin and C. Sabbah, and discussions with them were very fruitful. G.-M. Greuel and G. Pfister sharpened my view of moduli problems. M. Schulze and M. Rosellen made useful comments.

Of course, this book builds on the work of many people in singularity theory; Arnold, Givental, Looijenga, Malgrange, K. Saito, M. Saito, Scherk, O.P. Shcherbak, Slodowy, Steenbrink, Teissier, Varchenko, Wall, and many others. A good part of the book was written during a stay at the mathematics department of the University Paul Sabatier in Toulouse. I thank the department and especially J.-F. Mattei for their hospitality.

Bonn, July 2001 Claus Hertling