Part 1

Multiplication on the tangent bundle

Chapter 1

Introduction to part 1

An F-manifold is a complex manifold M such that each holomorphic tangent space $T_t M$, $t \in M$, is a commutative and associative algebra with unit element, and the multiplication varies in a specific way with the point $t \in M$. More precisely, it is a triple $(M, \circ, e)$ where $\circ$ is an $\mathcal{O}_M$-bilinear commutative and associative multiplication on the holomorphic tangent sheaf $\mathcal{T}_M$, e is a global unit field, and the multiplication satisfies the integrability condition

$$\mathrm{Lie}_{X\circ Y}(\circ) = X \circ \mathrm{Lie}_Y(\circ) + Y \circ \mathrm{Lie}_X(\circ) \tag{1.1}$$

for any two local vector fields X and Y in $\mathcal{T}_M$. This notion was first defined in [**HM**][**Man2**, I§5], motivated by Frobenius manifolds. Frobenius manifolds are F-manifolds.

Part 1 of this book is devoted to the local structure of F-manifolds. It turns out to be closely related to singularity theory and symplectic geometry. Discriminants and Lagrange maps play a key role.

In the short section 1.1 of this introduction the reader can experience some of the geometry of F-manifolds. We sketch a construction of 2-dimensional F-manifolds which shows how F-manifolds turn up 'in nature' and how they are related to discriminants. In section 1.2 we offer a fast track through the main notions and results of chapters 2 to 5.

In chapters 2 to 4 the general structure of F-manifolds is developed. In chapter 5 the most important classes of F-manifolds are discussed.

In chapter 2 F-manifolds are defined and some basic properties are established. One property shows that F-manifolds decompose locally in a nice way. Another one describes the relation to connections, metrics, and the potentiality condition of Frobenius manifolds.

In chapter 3 the relation to symplectic geometry and especially to Lagrange maps is discussed. This allows use to be made of Givental's paper [**Gi2**] on singular Lagrange varieties and their Lagrange maps.

Chapter 4 presents several notions and results, which are mostly motivated by corresponding notions and results in singularity theory. Most important are the discriminants and their geometry.

In chapter 5 F-manifolds from hypersurface singularities, boundary singularities, and Coxeter groups are discussed. In the case of Coxeter groups we extend some results of Givental [**Gi2**] and use them to prove a conjecture of Dubrovin about their Frobenius manifolds.

The reader should have the following background. There should be familiarity with the basic concepts of complex analytic geometry, including coherent sheaves and flatness. One reference is [**Fi**]. There should also be awareness of those notions from symplectic geometry which are treated in [**AGV1**, chapter 18] (canonical 1-form on the cotangent bundle, Lagrange fibration, Lagrange map, generating function). We recommend this reference. In chapter 5 some acquaintance with singularity theory makes the reading easier, but it is not necessary. Good references are [**AGV1**] and [**Lo2**].

1.1 First examples

To give the reader an idea of what F-manifolds look like and how they arise naturally, a construction of 2-dimensional F-manifolds is sketched. A systematic treatment is given in sections 4.1 and 4.2.

Let W be a finite Coxeter group of type $I_2(m)$, $m \geq 2$, acting on $\mathbb{R}^2$ and (by $\mathbb{C}$-linear extension) on $\mathbb{C}^2$. Then the ring $\mathbb{C}[x_1, x_2]^W \subset \mathbb{C}[x_1, x_2]$ of W-invariant polynomials is $\mathbb{C}[x_1, x_2]^W \cong \mathbb{C}[t_1, t_2]$ with 2 homogeneous generators t_1 and t_2 of degrees m and 2. Therefore the quotient space $\mathbb{C}^2/W =: M$ is isomorphic to $\mathbb{C}^2$ as an affine algebraic variety, and the vector field $e := \frac{\partial}{\partial t_1}$ is unique up to multiplication by a constant. The image in M of the union of the complexified reflection hyperplanes is the discriminant $\mathcal{D}$. We choose t_1 and t_2 such that it is given as $\mathcal{D} = \{t \in M \mid t_1^2 - \frac{4}{m^2} t_2^m = 0\}$.

For a point $t \in M$ with $t_2 \neq 0$, the pair $(e, \mathcal{D})$ gives rise to a multiplication on $T_t M$ in the following way, which is illustrated in figure 1.1.

The e-orbit through the point t intersects the discriminant at 2 points. We shift the tangent hyperplanes of $\mathcal{D}$ at these points with the flow of e to $T_t M$. We find that they are transverse to one another and to e. Therefore there are 2 unique vectors e_1 and e_2 in $T_t M$ which are tangent to these lines and satisfy $e = e_1 + e_2$. We define a multiplication on $T_t M$ by $e_i \circ e_j := \delta_{ij} e_i$. It is obviously commutative and associative, and e is the unit vector.

If we write this multiplication in terms of the coordinate fields $e := \frac{\partial}{\partial t_1}$ and $\frac{\partial}{\partial t_2}$, after some calculation we find $\frac{\partial}{\partial t_2} \circ \frac{\partial}{\partial t_2} = t_2^{m-2} \cdot \frac{\partial}{\partial t_1}$, and e is the

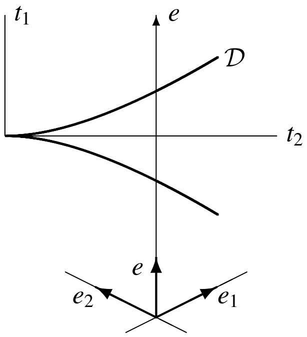

Figure 1.1

unit field. Therefore the multiplication extends holomorphically to the whole tangent bundle TM. One can show that it satisfies (1.1). The orbit space M is an F-manifold.

This construction of an F-manifold from a discriminant $\mathcal{D}$ and a transversal vector field e extends to higher dimensions (Corollary 4.6) and yields F-manifolds in many other cases, for example for all finite Coxeter groups (section 5.3).

1.2 Fast track through the results

The most notable (germs of) F-manifolds with many typical and some special properties are the base spaces of semiuniversal unfoldings of isolated hypersurface singularities and of boundary singularities (sections 5.1 and 5.2). Here the tangent space at each parameter is canonically isomorphic to the sum of the Jacobi algebras of the singularities above this parameter. Many of the general results on F-manifolds have been known in another guise in the hypersurface singularity case and all should be compared with it.

One reason why the integrability condition (1.1) is natural is the following: Let (M, p) be the germ of an F-manifold $(M, \circ, e)$. The algebra T_pM decomposes uniquely into a sum of (irreducible) local algebras which annihilate one another (Lemma 2.1). Now the integrability condition (1.1) ensures that this infinitesimal decomposition extends to a unique decomposition of the germ (M, p) into a product of germs of F-manifolds (Theorem 2.11).

If the multiplication at T_pM is semisimple, that is, if T_pM decomposes into 1-dimensional algebras, then this provides canonical coordinates $u_1, \ldots, u_n$ on (M, p) with $\frac{\partial}{\partial u_i} \circ \frac{\partial}{\partial u_j} = \delta_{ij} \frac{\partial}{\partial u_i}$. In fact, at points with semisimple multiplication the integrability condition (1.1) is equivalent to the existence of such canonical coordinates. In the hypersurface case, the decomposition of the germ (M, p)

for some parameter p is the unique decomposition into a product of base spaces of semiuniversal unfoldings of the singularities above p.

Another reason why (1.1) is natural is its relation to the potentiality of Frobenius manifolds. There exist F-manifolds such that not all tangent spaces are Frobenius algebras. They cannot be Frobenius manifolds. But if all tangent spaces are Frobenius algebras then the integrability condition (1.1) is related to a version of potentiality which requires a metric on M that is multiplication invariant, but not necessarily flat. See section 2.5 for details.

The most important geometric object which is attributed to an n-dimensional manifold M with multiplication $\circ$ on the tangent sheaf $\mathcal{T}_M$ and unit field e (with or without (1.1)) is the analytic spectrum $L := \text{Specan}(\mathcal{T}_M) \subset T^*M$ (see section 2.2). The projection $\pi : L \to M$ is flat and finite of degree n. The fibre $\pi^{-1}(p) \subset L$ above $p \in M$ consists of the set $\text{Hom}_{alg}(T_pM, \mathbb{C})$ of algebra homomorphisms from T_pM to $\mathbb{C}$; they correspond 1-1 to the irreducible subalgebras of $(T_pM, \circ)$ (see Lemma 2.1). The multiplication on $\mathcal{T}_M$ can be recovered from L, because the map

$$\mathbf{a} : \mathcal{T}_M \to \pi_*\mathcal{O}_L, \quad X \mapsto \alpha(\widetilde{X})|_L \tag{1.2}$$

is an isomorphism of $\mathcal{O}_M$-algebras; here $\widetilde{X}$ is any lift of X to T^*M and α is the canonical 1–form on T^*M. The values of the function $\mathbf{a}(X)$ on $\pi^{-1}(p)$ are the eigenvalues of $X\circ : T_pM \to T_pM$.

The analytic spectrum L is a reduced variety if and only if the multiplication is generically semisimple. Then the manifold with multiplication $(M, \circ, e)$ is called *massive*. Now, a third reason why the integrability condition (1.1) is natural is this: Suppose that $(M, \circ, e)$ is a manifold with generically semisimple multiplication. Then $L \subset T^*M$ is a Lagrange variety if and only if $(M, \circ, e)$ is a massive F-manifold (Theorem 3.2).

The main body of part 1 is devoted to the study of germs of massive F-manifolds at points where the multiplication is not semisimple.

We will make use of the theory of singular Lagrange varieties and their Lagrange maps, which has been worked out by Givental in [**Gi2**]. In fact, the notion of an irreducible germ (with respect to the above decomposition) of a massive F-manifold is equivalent to Givental's notion of a miniversal germ of a flat Lagrange map (Theorem 3.16). Via this equivalence Givental's paper contains many results on massive F-manifolds and will be extremely useful.

Locally the canonical 1–form α on T^*M can be integrated on the analytic spectrum L of a massive F-manifold $(M, \circ, e)$ to a *generating function* $F : L \to \mathbb{C}$ which is continuous on L and holomorphic on L_{reg}. It depends on a property of L, which is weaker than normality or maximality of the complex structure of L, whether F is holomorphic on L (see section 3.2).

If F is holomorphic on L then it corresponds via (1.2) to an *Euler field* $E = \mathbf{a}^{-1}(F)$ of weight 1, that is, a vector field on M with $\mathrm{Lie}_E(\circ) = \circ$ (Theorem 3.3).

In any case, a generating function $F : L \to \mathbb{C}$ gives rise to a Lyashko–Looijenga map $\Lambda : M \to \mathbb{C}^n$ (see sections 3.3 and 3.5) and a discriminant $\mathcal{D} = \pi(F^{-1}(0)) \subset M$.

If F is holomorphic and an Euler field $E = \mathbf{a}^{-1}(F)$ exists then the discriminant $\mathcal{D}$ is the hypersurface of points where the multiplication with E is not invertible. Then it is a free divisor with logarithmic fields $\mathrm{Der}_M(\log \mathcal{D}) = E \circ \mathcal{T}_M$ (Theorem 4.9). This generalizes a result of K. Saito for the hypersurface case.

From the unit field e and a discriminant $\mathcal{D} \subset M$ one can reconstruct everything. One can read off the multiplication on TM in a very nice elementary way (Corollary 4.6 and section 1.1): The e-orbit of a generic point $p \in M$ intersects $\mathcal{D}$ at n points. One shifts the n tangent hyperplanes with the flow of e to T_pM. Then there exist unique vectors $e_1(p), \ldots, e_n(p) \in T_pM$ such that $\sum_{i=1}^n e_i(p) = e(p)$ and $\sum_{i=1}^n \mathbb{C} \cdot e_i(p) = T_pM$ and such that the subspaces $\sum_{i \neq j} \mathbb{C} \cdot e_i(p)$, $j = 1, \ldots, n$, are the shifted hyperplanes. The multiplication on T_pM is given by $e_i(p) \circ e_j(p) = \delta_{ij} e_i(p)$.

In the case of hypersurface singularities and boundary singularities, the classical discriminant in the base space of a semiuniversal unfolding is such a discriminant. The critical set C in the total space of the unfolding is canonically isomorphic to the analytic spectrum L; this isomorphism identifies the map $\mathbf{a}$ in (1.2) with a Kodaira–Spencer map $\mathbf{a}_C : \mathcal{T}_M \to (\pi_C)_* \mathcal{O}_C$ and a generating function $F : L \to \mathbb{C}$ with the restriction of the unfolding function to the critical set C. This Kodaira–Spencer map $\mathbf{a}_C$ is the source of the multiplication on $\mathcal{T}_M$ in the hypersurface singularity case. The multiplication on $\mathcal{T}_M$ had first been defined in this way by K. Saito.

Critical set and analytic spectrum are smooth in the hypersurface singularity case. By the work of Arnold and Hörmander on Lagrange maps and singularities an excellent correspondence holds (Theorem 5.6): each irreducible germ of a massive F-manifold with smooth analytic spectrum comes from an isolated hypersurface singularity, and this singularity is unique up to stable right equivalence.

By the work of Nguyen huu Duc and Nguyen tien Dai the same correspondence holds for boundary singularities and irreducible germs of massive F-manifolds whose analytic spectrum is isomorphic to $(\mathbb{C}^{n-1}, 0) \times (\{(x, y) \in \mathbb{C}^2 \mid xy = 0\}, 0)$ with ordered components (Theorem 5.14).

The complex orbit space $M := \mathbb{C}^n / W \cong \mathbb{C}^n$ of a finite irreducible Coxeter group W carries an (up to some rescaling) canonical structure of a massive F-manifold: A generating system $P_1, \ldots, P_n$ of W-invariant homogeneous polynomials induces coordinates $t_1, \ldots, t_n$ on M. Precisely one polynomial, e.g. P_1, has highest degree. The field $\frac{\partial}{\partial t_1}$ is up to a scalar independent of any choices. This field $e := \frac{\partial}{\partial t_1}$ as the unit field and the classical discriminant $\mathcal{D} \subset M$, the image of the reflection hyperplanes, determine in the elementary way described above the structure of a massive F-manifold. This follows from [**Du2**][**Du3**, Lecture 4] as well as from [**Gi2**, Theorem 14].

Dubrovin established the structure of a Frobenius manifold on the complex orbit space $M = \mathbb{C}^n/W$, with this multiplication, with K. Saito's flat metric on M, and with a canonical Euler field with positive weights (see Theorem 5.23). At the same place he conjectured that these Frobenius manifolds and their products are (up to some well-understood rescalings) the only massive Frobenius manifolds with an Euler field with positive weights. We will prove this conjecture (Theorem 5.25).

Crucial for the proof is Givental's result [**Gi2**, Theorem 14]. It characterizes the germs $(M, 0)$ of F-manifolds of irreducible Coxeter groups by geometric properties (see Theorem 5.21). We obtain from it the following intermediate result (Theorem 5.20): An irreducible germ (M, p) of a *simple* F-manifold such that T_pM is a Frobenius algebra is isomorphic to the germ at 0 of the F-manifold of an irreducible Coxeter group.

A massive F-manifold $(M, \circ, e)$ is called *simple* if the germs (M, p), $p \in M$, of F-manifolds are contained in finitely many isomorphism classes. Theorem 5.20 complements in a nice way the relation of irreducible Coxeter groups to the simple hypersurface singularities A_n, D_n, E_n and the simple boundary singularities B_n, C_n, F_4.

In dimensions 1 and 2, up to isomorphism all the irreducible germs of massive F-manifolds come from the irreducible Coxeter groups A_1 and $I_2(m)$ $(m \geq 3)$ with $I_2(3) = A_2,\ I_2(4) = B_2,\ I_2(5) =: H_2,\ I_2(6) = G_2$. But already in dimension 3 the classification is vast (see section 5.5).

Chapter 2

Definition and first properties of F-manifolds

An F-manifold is a manifold with a multiplication on the tangent bundle which satisfies a certain integrability condition. It is defined in section 2.3. Sections 2.4 and 2.5 give two reasons why this is a good notion. In section 2.4 it is shown that germs of F-manifolds decompose in a nice way. In section 2.5 the relation to connections and metrics is discussed. It turns out that the integrability condition is part of the potentiality condition for Frobenius manifolds. Therefore Frobenius manifolds are F-manifolds.

Section 2.1 is a self-contained elementary account of the structure of finite dimensional algebras in general (e.g. the tangent spaces of an F-manifold) and Frobenius algebras in particular. Section 2.2 discusses vector bundles with multiplication. There the caustic and the analytic spectrum are defined, two notions which are important for F-manifolds.

2.1 Finite-dimensional algebras

In this section $(Q, \circ, e)$ is a $\mathbb{C}$-algebra of finite dimension (≥ 1) with commutative and associative multiplication and with unit e. The next lemma gives precise information on the decomposition of Q into irreducible algebras. The statements are well known and elementary. They can be deduced directly in the given order or from more general results in commutative algebra (Q is an Artin algebra). Algebra homomorphisms are always supposed to map the unit to the unit.

Lemma 2.1 *Let $(Q, \circ, e)$ be as above. As the endomorphisms $x\circ : Q \to Q$, $x \in Q$, commute, there is a unique simultaneous decomposition $Q = \bigoplus_{k=1}^{l} Q_k$ into generalized eigenspaces Q_k (with $\dim_{\mathbb{C}} Q_k \geq 1$). Define $e_k \in Q_k$ by $e = \sum_{k=1}^{l} e_k$. Then*

(i) One has $Q_j \circ Q_k = 0$ for $j \neq k$; also $e_k \neq 0$ and $e_j \circ e_k = \delta_{jk} e_k$; the element e_k is the unit of the algebra $Q_k = e_k \circ Q$.

(ii) *The function* $\lambda_k : Q \to \mathbb{C}$ *which associates to* $x \in Q$ *the eigenvalue of* $x \circ$ *on* Q_k *is an algebra homomorphism;* $\lambda_j \neq \lambda_k$ *for* $j \neq k$.
(iii) *The algebra* $(Q_k, \circ, e_k)$ *is an irreducible and a local algebra with maximal ideal* $\mathbf{m}_k = Q_k \cap \ker(\lambda_k)$.
(iv) *The subsets* $\ker(\lambda_k) = \mathbf{m}_k \oplus \bigoplus_{j \neq k} Q_j$, $k = 1, \ldots, l$, *are the maximal ideals of the algebra* Q*; the complement* $Q - \bigcup_k \ker(\lambda_k)$ *is the group of invertible elements of* Q.
(v) *The set* $\{\lambda_1, \ldots, \lambda_l\} = \mathrm{Hom}_{\mathbb{C}-alg}(Q, \mathbb{C})$.
(vi) *The localization* $Q_{\ker(\lambda_k)}$ *is isomorphic to* Q_k.

We call this decomposition the *eigenspace decomposition* of $(Q, \circ, e)$. The set $L := \{\lambda_1, \ldots, \lambda_l\} \subset Q^*$ carries a natural complex structure $\mathcal{O}_L$ such that $\mathcal{O}_L(L) \cong Q$ and $\mathcal{O}_{L,\lambda_k} \cong Q_k$. More details on this will be given in section 2.2.

The algebra (or its multiplication) is called *semisimple* if Q decomposes into 1-dimensional subspaces, $Q \cong \bigoplus_{k=1}^{\dim Q} Q_k = \bigoplus_{k=1}^{\dim Q} \mathbb{C} \cdot e_k$.

An irreducible algebra $Q = \mathbb{C} \cdot e \oplus \mathbf{m}$ with maximal ideal $\mathbf{m}$ is a *Gorenstein ring* if the socle $\mathrm{Ann}_Q(\mathbf{m})$ has dimension 1.

An algebra $Q = \bigoplus_{k=1}^{l} Q_k$ is a *Frobenius algebra* if each irreducible subalgebra is a Gorenstein ring (cf. for example [**Kun**]).

The next (also well known) lemma gives equivalent conditions and additional information. Note that this classical definition of a Frobenius algebra is slightly weaker than Dubrovin's: he calls an algebra $(Q, \circ, e)$ together with a *fixed* bilinear form g as in Lemma 2.2 (a) (iii) a Frobenius algebra.

Lemma 2.2 *(a) The following conditions are equivalent.*

(i) *The algebra* $(Q, \circ, e)$ *is a Frobenius algebra.*
(ii) *As a* Q*-module* $\mathrm{Hom}(Q, \mathbb{C}) \cong Q$.
(iii) *There exists a bilinear form* $g : Q \times Q \to \mathbb{C}$ *which is symmetric, non-degenerate and multiplication invariant, i.e.* $g(a \circ b, c) = g(a, b \circ c)$.

(b) Let $Q = \bigoplus_{k=1}^{l} Q_k$ *be a Frobenius algebra and* $Q_k = \mathbb{C} \cdot e_k \oplus \mathbf{m}_k$. *The generators of* $\mathrm{Hom}(Q, \mathbb{C})$ *as a* Q*-module are the linear forms* $f : Q \to \mathbb{C}$ *with* $f(\mathrm{Ann}_{Q_k}(\mathbf{m}_k)) = \mathbb{C}$ *for all* k.

One obtains a 1-1 correspondence between these linear forms and the bilinear forms g *as in (a) (iii) by putting* $g(x, y) := f(x \circ y)$.

Proof: (a) Any of the conditions (i), (ii), (iii) in (a) is satisfied for Q if and only if it is satisfied for each irreducible subalgebra Q_k. One checks this with $Q_j \circ Q_k = 0$ for $j \neq k$. So we may suppose that Q is irreducible.

(i) $\iff$ (ii) A linear form $f \in \mathrm{Hom}(Q, \mathbb{C})$ generates $\mathrm{Hom}(Q, \mathbb{C})$ as a Q-module if and only if the linear form $(x \mapsto f(y \circ x))$ is nontrivial for any $y \in Q - \{0\}$, that is, if and only if $f(y \circ Q) = \mathbb{C}$ for any $y \in Q - \{0\}$.

The socle $\mathrm{Ann}_Q(\mathbf{m})$ is the set of the common eigenvectors of all endomorphisms $x\circ : Q \to Q, x \in Q$. If $\dim \mathrm{Ann}_Q(\mathbf{m}) \geq 2$ then for any linear form f an element $y \in (\ker f \cap \mathrm{Ann}_Q(\mathbf{m})) - \{0\}$ satisfies $y \circ Q = \mathbb{C} \cdot y$ and $f(y \circ Q) = 0$; so f does not generate $\mathrm{Hom}(Q, \mathbb{C})$. If $\dim \mathrm{Ann}_Q(\mathbf{m}) = 1$ then it is contained in any nontrivial ideal, because any such ideal contains a common eigenvector of all endomorphisms. The set $y \circ Q$ for $y \in Q - \{0\}$ is an ideal. So, then a linear form f with $f(\mathrm{Ann}_Q(\mathbf{m})) = \mathbb{C}$ generates $\mathrm{Hom}(Q, \mathbb{C})$ as a Q-module.

(i) $\Rightarrow$ (iii) Choose any linear form f with $f(\mathrm{Ann}_Q(\mathbf{m})) = \mathbb{C}$ and define g by $g(x, y) := f(x \circ y)$. It remains to show that g is nondegenerate. But for any $x \in Q - \{0\}$ there exists a $y \in Q$ with $\mathbb{C} \cdot x \circ y = \mathrm{Ann}_Q(\mathbf{m})$, because $\mathrm{Ann}_Q(\mathbf{m})$ is contained in the ideal $x \circ Q$.

(iii) $\Rightarrow$ (i) The equalities $g(\mathbf{m}, \mathrm{Ann}_Q(\mathbf{m})) = g(e, \mathbf{m} \circ \mathrm{Ann}_Q(\mathbf{m})) = g(e, 0) = 0$ imply $\dim \mathrm{Ann}_Q(\mathbf{m}) = 1$.

(b) Starting with a bilinear form g, the corresponding linear form f is given by $f(x) = g(x, e)$. The rest is clear from the preceding discussion. □

The semisimple algebra $Q \cong \bigoplus_{k=1}^{\dim Q} \mathbb{C} \cdot e_k$ is a Frobenius algebra.

A classical result is that the complete intersections $\mathcal{O}_{\mathbb{C}^m,0}/(f_1, \ldots, f_m)$ are Gorenstein. But there are other Gorenstein algebras, e.g. $\mathbb{C}\{x, y, z\}/(x^2, y^2, xz, yz, xy - z^2)$ is Gorenstein, but not a complete intersection.

Finally, in the next section vector bundles with multiplication will be considered. Condition (ii) of Lemma 2.2 (a) shows that there the points whose fibres are Frobenius algebras form an open set in the base.

2.2 Vector bundles with multiplication

Now we consider a holomorphic vector bundle $Q \to M$ on a complex manifold M with multiplication on the fibres: The sheaf $\mathcal{Q}$ of holomorphic sections of the bundle $Q \to M$ is equipped with an $\mathcal{O}_M$-bilinear commutative and associative multiplication $\circ$ and with a global unit section e.

The set $\bigcup_{p \in M} \mathrm{Hom}_{\mathbb{C}-alg}(Q(p), \mathbb{C})$ of algebra homomorphisms from the single fibres $Q(p)$ to $\mathbb{C}$ (which map the unit to $1 \in \mathbb{C}$) is a subset of the dual bundle Q^* and has a natural complex structure. It is the analytic spectrum $\mathrm{Specan}(\mathcal{Q})$. We sketch the definition ([**Hou**, ch. 3], also [**Fi**, 1.14]):

The $\mathcal{O}_M$-sheaf $\mathrm{Sym}_{\mathcal{O}_M} \mathcal{Q}$ can be identified with the $\mathcal{O}_M$-sheaf of holomorphic functions on Q^* which are polynomial in the fibres. The canonical $\mathcal{O}_M$-algebra homomorphism $\mathrm{Sym}_{\mathcal{O}_M} \mathcal{Q} \to \mathcal{Q}$ which maps the multiplication in $\mathrm{Sym}_{\mathcal{O}_M} \mathcal{Q}$