1 Introduction

1.1 What is Vortex Sound?

Vortex sound is the sound produced as a by-product of unsteady fluid motions (Fig. 1.1.1). It is part of the more general subject of aerodynamic sound. The modern theory of aerodynamic sound was pioneered by James Lighthill in the early 1950s. Lighthill (1952) wanted to understand the mechanisms of noise generation by the jet engines of new passenger jet aircraft that were then about to enter service. However, it is now widely recognized that *any* mechanism that produces sound can actually be formulated as a problem of aerodynamic sound. Thus, apart from the high speed turbulent jet – which may be regarded as a distribution of intense turbulence velocity fluctuations that generate sound by converting a tiny fraction of the jet *rotational* kinetic energy into the longitudinal waves that constitute sound – colliding solid bodies, aeroengine rotor blades, vibrating surfaces, complex fluid–structure interactions in the larynx (responsible for speech), musical instruments, conventional loudspeakers, crackling paper, explosions, combustion and combustion instabilities in rockets, and so forth all fall within the theory of aerodynamic sound in its broadest sense.

In this book we shall consider principally the production of sound by unsteady motions of a fluid. Any fluid that possesses intrinsic kinetic energy, that is, energy not directly attributable to a moving boundary (which is largely withdrawn from the fluid when the boundary motion ceases), must possess *vorticity*. We shall see that in a certain sense and for a vast number of flows vorticity may be regarded as the ultimate source of the sound generated by the flow. Our objective, therefore, is to simplify the general aerodynamic sound problem to obtain a thorough understanding of how this happens, and of how the sound can be estimated quantitatively.

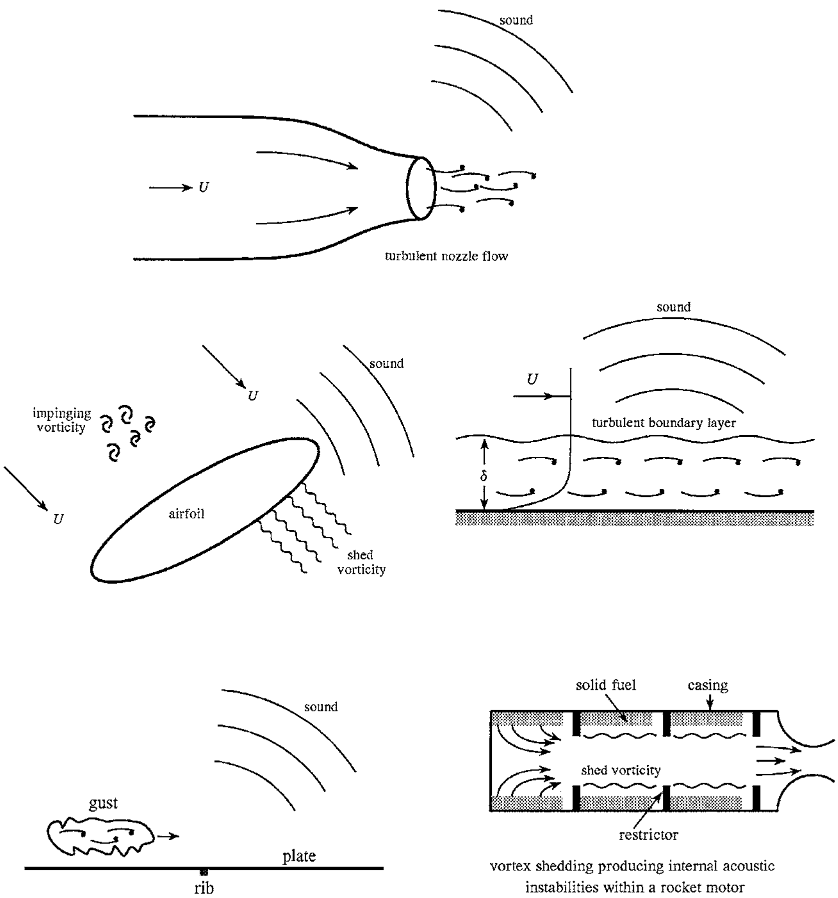

Fig. 1.1.1. Typical vortex sound problems.

1.2 Equations of Motion of a Fluid

At time t and position $\mathbf{x} = (x_1, x_2, x_3)$, the state of a fluid is defined when the velocity $\mathbf{v}$ and any two thermodynamic variables are specified. Five scalar equations are therefore required to determine the motion. These equations are statements of the conservation of mass, momentum, and energy.

1.2.1 Equation of Continuity

Conservation of mass requires the rate of increase of the fluid mass within a fixed region of space V to be equal to the net influx due to convection across the boundaries of V. The velocity $\mathbf{v}$ and the fluid density ρ must therefore satisfy the equation of continuity, which has the following equivalent forms

$$\left.\begin{aligned}\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\mathbf{v}) = 0,\\ \frac{1}{\rho}\frac{D\rho}{Dt} + \operatorname{div}\mathbf{v} = 0,\\ \operatorname{div}\mathbf{v} = \rho\frac{D}{Dt}\left(\frac{1}{\rho}\right)\end{aligned}\right\}, \tag{1.2.1}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla \equiv \frac{\partial}{\partial t} + v_j\frac{\partial}{\partial x_j} \tag{1.2.2}$$

is the *material derivative;* the repeated suffix j implies summation over $j = 1, 2, 3$. The last of Equations (1.2.1) states that $\operatorname{div}\mathbf{v}$ is equal to the rate of change of fluid volume per unit volume following the motion of the fluid. For an incompressible fluid this is zero, i.e., $\operatorname{div}\mathbf{v} = 0$.

1.2.2 Momentum Equation

The momentum equation is also called the *Navier–Stokes equation;* it expresses the rate of change of momentum of a fluid particle in terms of the pressure p, the **viscous** or frictional force, and body forces $\mathbf{F}$ per unit volume. We consider only *Stokesian fluids* (most liquids and monatomic gases, but also a good approximation in air for calculating the frictional drag at a solid boundary) for which the principal frictional forces are expressed in terms of the shear coefficient of viscosity η, which we shall invariably assume to be constant. Then the momentum equation is

$$\rho\frac{D\mathbf{v}}{Dt} = -\nabla p + \eta\left(\nabla^2\mathbf{v} + \frac{1}{3}\nabla(\operatorname{div}\mathbf{v})\right) + \mathbf{F}. \tag{1.2.3}$$

Values of ρ, η and $\nu = \eta/\rho$ (the 'kinematic' viscosity) for air and water at 10 °C and one atmosphere pressure are given in the Table 1.2.1:

Table 1.2.1. *Density and viscosity*

	ρ, kg/m^3	η, kg/ms	ν, m^2/s
Air	1.23	1.764×10^{-5}	1.433×10^{-5}
Water	1000	1.284×10^{-3}	1.284×10^{-6}

1.2.3 Energy Equation

This equation must be used in its full generality in problems where energy is transferred by heat conduction, where frictional dissipation of sound is occurring, when shock waves are formed by highly nonlinear events, or when sound is being generated by combustion and other heat sources. For our purposes it will usually be sufficient to suppose the flow to be *homentropic;* namely, the specific **entropy** s of the fluid is uniform and constant throughout the fluid, so that the energy equation becomes

$$s = \text{constant.} \tag{1.2.4}$$

We may then assume that the pressure and density are related by an equation of the form

$$p = p(\rho, s), \quad s = \text{constant.} \tag{1.2.5}$$

This equation will be satisfied by both the mean (undisturbed) and unsteady components of the flow. Thus, for an ideal gas

$$p = \text{constant} \times \rho^{\gamma}, \quad \gamma = \text{ratio of specific heats.} \tag{1.2.6}$$

1.3 Equation of Linear Acoustics

The intensity of a sound pressure p in air is usually measured on a decibel scale by the quantity

$$20 \times \log_{10}\left(\frac{|p|}{p_{\text{ref}}}\right),$$

where the reference pressure $p_{\text{ref}} = 2 \times 10^{-5}$ N/m^2. Thus, $p = p_0 \equiv 1$ atmosphere ($= 10^5$ N/m^2) is equivalent to 194 dB. A very loud sound $\sim$120 dB corresponds to

$$\frac{p}{p_0} \approx \frac{2 \times 10^{-5}}{10^5} \times 10^{\left(\frac{120}{20}\right)} = 2 \times 10^{-4} \ll 1.$$

Similarly, for a 'deafening' sound of 160 dB, $p/p_0 \sim 0.02$. This corresponds to a pressure of about 0.3 lbs/in^2 and is loud enough for nonlinear effects to begin to be important.

The passage of a sound wave in the form of a pressure fluctuation is, of course, accompanied by a back-and-forth motion of the fluid at the *acoustic* *particle velocity* v, say. We shall see later that

$$\text{acoustic particle velocity} \approx \frac{\text{acoustic pressure}}{\text{mean density} \times \text{speed of sound}}.$$

In air the speed of sound is about 340 m/sec. Thus, at 120 dB $v \sim 5$ cm/sec; at 160 dB $v \sim 5$ m/sec.

In most applications the acoustic amplitude is very small relative to the mean pressure p_0, and sound propagation may be studied by linearizing the equations. To do this we shall first consider sound propagating in a *stationary* inviscid fluid of mean pressure p_0 and density ρ_0; let the departures of the pressure and density from these mean values be denoted by p', ρ', where $p'/p_0 \ll 1$, $\rho'/\rho_0 \ll 1$. The linearized momentum equation (1.2.3) becomes

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} + \nabla p' = \mathbf{F}. \tag{1.3.1}$$

Before linearizing the continuity equation (1.2.1), we introduce an artificial generalization by inserting a **volume source** distribution $q(\mathbf{x}, t)$ on the right-hand side

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \operatorname{div} \mathbf{v} = q, \tag{1.3.2}$$

where q is the rate of increase of fluid volume per unit volume of the fluid, and might represent, for example, the effect of volume pulsations of a small body in the fluid. The linearized equation is then

$$\frac{1}{\rho_0}\frac{\partial \rho'}{\partial t} + \operatorname{div} \mathbf{v} = q. \tag{1.3.3}$$

Now eliminate $\mathbf{v}$ between (1.3.1) and (1.3.3):

$$\frac{\partial^2 \rho'}{\partial t^2} - \nabla^2 p' = \rho_0 \frac{\partial q}{\partial t} - \operatorname{div} \mathbf{F}. \tag{1.3.4}$$

An equation determining the pressure p' alone in terms of q and $\mathbf{F}$ is obtained by invoking the homentropic relation (1.2.5). In the undisturbed and disturbed states we have

$$p_0 = p(\rho_0, s), \qquad p_0 + p' = p(\rho_0 + \rho', s) \approx p(\rho_0, s) + \left(\frac{\partial p}{\partial \rho}(\rho, s)\right)_0 \rho', \quad s = \text{constant}. \tag{1.3.5}$$

The derivative is evaluated at the undisturbed values of the pressure and density (p_0, ρ_0). It has the dimensions of velocity^2, and its square root defines the *speed of sound*

$$c_0 = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s}, \tag{1.3.6}$$

where the derivative is taken with the entropy s held fixed at its value in the undisturbed fluid. The implication is that losses due to heat transfer between neighboring fluid particles by viscous and thermal diffusion are neglected during the passage of a sound wave (i.e., that the motion of a fluid particle is *adiabatic*).

From (1.3.5): $\rho' = p'/c_0^2$. Hence, substituting for ρ' in (1.3.4), we obtain

$$\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)p = \rho_0\frac{\partial q}{\partial t} - \operatorname{div}\mathbf{F}, \tag{1.3.7}$$

where the prime ($'$) on the acoustic pressure has been discarded. This equation governs the production of sound waves by the volume source q and the force $\mathbf{F}$. When these terms are absent the equation describes sound propagation from sources on the boundaries of the fluid, such as the vibrating cone of a loudspeaker.

The volume source q and the body force $\mathbf{F}$ would never appear in a complete description of sound generation within a fluid. They are introduced only when we *think* we understand how to model the real sources of sound in terms of volume sources and forces. In general this can be a dangerous procedure because, as we shall see, small errors in specifying the sources of sound in a fluid can lead to very large errors in the predicted sound. This is because only a tiny fraction of the available energy of a vibrating fluid or structure actually radiates away as sound.

When $\mathbf{F} = \mathbf{0}$, Equation (1.3.1) implies the existence of a velocity potential φ such that $\mathbf{v} = \nabla\varphi$, in terms of which the perturbation pressure is given by

$$p = -\rho_0\frac{\partial \varphi}{\partial t}. \tag{1.3.8}$$

It follows from this and (1.3.7) (with $\mathbf{F} = \mathbf{0}$) that the velocity potential is the solution of

$$\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\varphi = -q(\mathbf{x}, t). \tag{1.3.9}$$

This is the wave equation of classical acoustics.

Table 1.3.1. *Speed of sound and acoustic wavelength*

	c_0				λ at 1 kHz	
	m/s	ft/s	km/h	mi/h	m	ft
Air	340	1100	1225	750	0.3	1
Water	1500	5000	5400	3400	1.5	5

For future reference, Table 1.3.1 lists the approximate speeds of sound in air and in water, and the corresponding acoustic wavelength λ at a frequency of 1 kHz (sound of frequency f has wavelength $\lambda = c_0/f$).

1.4 The Special Case of an Incompressible Fluid

Small (adiabatic) pressure and density perturbations δp and $\delta\rho$ satisfy

$$\frac{\delta p}{\delta \rho} \approx c_0^2.$$

In an incompressible fluid the pressure can change by the action of external forces (moving boundaries, etc.), but the density must remain fixed. Thus, $c_0 = \infty$, and Equation (1.3.9) reduces to

$$\nabla^2 \varphi = q(\mathbf{x}, t). \tag{1.4.1}$$

1.4.1 Pulsating Sphere

Consider the motion produced by small amplitude radial pulsations of a sphere of mean radius a. Let the center of the sphere be at the origin, and let its normal velocity be $v_n(t)$. There are no sources within the fluid, so that $q \equiv 0$. Therefore,

$$\left.\begin{aligned} \nabla^2\varphi &= 0, & r &> a, \\ \partial\varphi/\partial r &= v_n(t), & r &= a \end{aligned}\right\} \quad \text{where } r = |\mathbf{x}|.$$

The motion is obviously radially symmetric, so that

$$\nabla^2\varphi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\varphi = 0, \quad r > a.$$

Hence,

$$\varphi = \frac{A}{r} + B,$$

where $A \equiv A(t)$ and $B \equiv B(t)$ are functions of t. $B(t)$ can be discarded because the pressure fluctuations ($\sim -\rho_0 \partial\varphi/\partial t$) must vanish as $r \to \infty$. Applying the condition $\partial\varphi/\partial r = v_n$ at $r = a$, we then find

$$\varphi = -\frac{a^2 v_n(t)}{r}, \quad r > a. \tag{1.4.2}$$

Thus, the pressure

$$p = -\rho_0 \frac{\partial\varphi}{\partial t} = \rho_0 \frac{a^2}{r}\frac{dv_n}{dt}(t)$$

decays as $1/r$ with distance from the sphere, and exhibits the unphysical characteristic of changing instantaneously everywhere when dv_n/dt changes its value. For any time t, the volume flux $q(t)$ of fluid is the same across any closed surface enclosing the sphere. Evaluating it for any sphere S of radius $r > a$, as shown in Fig. 1.4.1, we find

$$q(t) = \oint_S \nabla\varphi \cdot d\mathbf{S} = 4\pi a^2 v_n(t),$$

and we may also write

$$\varphi = \frac{-q(t)}{4\pi r}, \quad r > a. \tag{1.4.3}$$

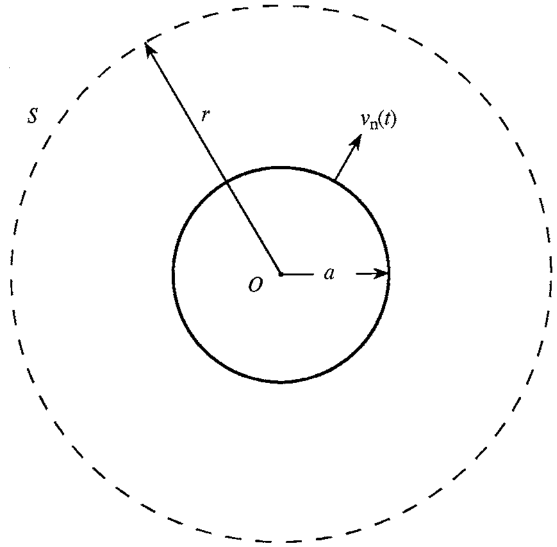

Fig. 1.4.1.

1.4.2 Point Source

The incompressible motion generated by a *volume* point source of strength $q(t)$ at the origin is the solution of

$$\nabla^2\varphi = q(t)\delta(\mathbf{x}), \quad \text{where } \delta(\mathbf{x}) = \delta(x_1)\delta(x_2)\delta(x_3). \tag{1.4.4}$$

The solution must be radially symmetric and given by

$$\varphi = \frac{A}{r} \quad \text{for } r > 0. \tag{1.4.5}$$

To find A, we integrate (1.4.4) over the interior of a sphere of radius $r = R > 0$, and use the divergence theorem $\int_{r<R} \nabla^2\varphi \, d^3\mathbf{x} = \oint_S \nabla\varphi \cdot d\mathbf{S}$, where S is the surface of the sphere. Then

$$\oint_S \nabla\varphi \cdot d\mathbf{S} \equiv \left(\frac{-A}{R^2}\right) \times (4\pi R^2) = q(t).$$

Hence, $A = -q(t)/4\pi$ and $\varphi = -q(t)/4\pi r$, which agrees with the solution (1.4.3) for the sphere with the same volume outflow in the region $r > a =$ radius of the sphere. This indicates that when we are interested in modelling the effect of a pulsating sphere at large distances $r \gg a$, it is permissible to replace the sphere by a point source (a monopole) of the same strength $q(t) =$ rate of change of the volume of the sphere. This conclusion is valid for any pulsating body, not just a sphere. However, it is not necessarily a good model (especially when we come to examine the production of *sound* by a pulsating body) in the presence of a *mean fluid flow* past the sphere.

The Solution (1.4.5) for the point source is strictly valid only for $r > 0$, where it satisfies $\nabla^2\varphi = 0$. What happens as $r \to 0$, where its value is actually undefined? To answer this question, we write the solution in the form

$$\varphi = \lim_{\epsilon \to 0} \frac{-q(t)}{4\pi(r^2+\epsilon^2)^{\frac{1}{2}}}, \quad \epsilon > 0, \quad \text{in which case } \nabla^2\varphi = \lim_{\epsilon\to 0} \frac{3\epsilon^2 q(t)}{4\pi(r^2+\epsilon^2)^{\frac{5}{2}}}.$$

The last limit is just equal to $q(t)\delta(\mathbf{x})$. Indeed when ϵ is small $3\epsilon^2/4\pi(r^2+\epsilon^2)^{\frac{5}{2}}$ is also small except close to $r = 0$, where it attains a large maximum $\sim 3/4\pi\epsilon^3$. Therefore, for any smoothly varying test function $f(\mathbf{x})$ and any volume V

enclosing the origin

$$\lim_{\epsilon\to 0}\int_V \frac{3\epsilon^2 f(\mathbf{x})\,d^3\mathbf{x}}{4\pi(r^2+\epsilon^2)^{\frac{5}{2}}} = f(\mathbf{0})\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\frac{3\epsilon^2\,d^3\mathbf{x}}{4\pi(r^2+\epsilon^2)^{\frac{5}{2}}}$$
$$= f(\mathbf{0})\int_0^{\infty}\frac{3\epsilon^2 r^2\,dr}{(r^2+\epsilon^2)^{\frac{5}{2}}} = f(\mathbf{0}),$$

where the value of the last integral is independent of ϵ. This is the defining property of the three-dimensional δ function.

Thus, the correct interpretation of the solution

$$\varphi = \frac{-1}{4\pi r} \quad \text{of} \quad \nabla^2\varphi = \delta(\mathbf{x}) \tag{1.4.6}$$

for a *unit* point source ($q = 1$) is

$$\frac{-1}{4\pi r} = \lim_{\epsilon\to 0}\frac{-1}{4\pi(r^2+\epsilon^2)^{\frac{1}{2}}}, \quad r \geq 0, \tag{1.4.7}$$

where

$$\nabla^2\left(\frac{-1}{4\pi r}\right) = \lim_{\epsilon\to 0}\nabla^2\left(\frac{-1}{4\pi(r^2+\epsilon^2)^{\frac{1}{2}}}\right) = \lim_{\epsilon\to 0}\frac{3\epsilon^2}{4\pi(r^2+\epsilon^2)^{\frac{5}{2}}} = \delta(\mathbf{x}). \tag{1.4.8}$$

1.5 Sound Produced by an Impulsive Point Source

The sound generated by the unit, impulsive point source $\delta(\mathbf{x})\delta(t)$ is the solution of

$$\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\varphi = \delta(\mathbf{x})\delta(t). \tag{1.5.1}$$

The source exists only for an infinitesimal instant of time at $t = 0$; therefore at earlier times $\varphi(\mathbf{x}, t) = 0$ everywhere.

It is evident that the solution is radially symmetric, and that for $r = |\mathbf{x}| > 0$ we have to solve

$$\frac{1}{c_0^2}\frac{\partial^2\varphi}{\partial t^2} - \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\varphi = 0, \quad r > 0. \tag{1.5.2}$$

The identity

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\varphi \equiv \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\varphi) \tag{1.5.3}$$