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References to the list of references are not made, and names associated with concepts, such as Bohr radius, Bessel functions, Schrödinger equation, etc., are not mentioned. As regards N. Fröman and P. O. Fröman references are made only to Chapter 1.

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