

## GRAVITY–CAPILLARY FREE-SURFACE FLOWS

Free-surface problems occur in many aspects of science and of everyday life, for example in the waves on a beach, bubbles rising in a glass of champagne, melting ice, pouring flows from a container and sails billowing in the wind. Consequently, the theory of gravity–capillary free-surface flows continues to be a fertile field of research in applied mathematics and engineering.

Concentrating on applications arising from fluid dynamics, Vanden-Broeck draws upon his years of experience in the field to address the many challenges involved in attempting to describe such flows mathematically. Whilst careful numerical techniques are implemented to solve the basic equations, an emphasis is placed upon the reader developing a deep understanding of the structure of the resulting solutions. The author also reviews relevant concepts in fluid mechanics to enable readers from other scientific fields to develop a working knowledge of free-boundary problems.

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GRAVITY–CAPILLARY  
FREE-SURFACE  
FLOWS

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To Mirna and Ada

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## Preface

This book is concerned with the theory of gravity–capillary free-surface flows. Free-surface flows are flows bounded by surfaces that have to be found as part of the solution. A canonical example is that of waves propagating on a water surface, the latter in this case being the free surface.

Many other examples of free-surface flows are considered in the book (cavitating flows, free-surface flows generated by moving disturbances, rising bubbles etc.). I hope to convince the reader of the beauty of such problems and to elucidate some mathematical challenges faced when solving them. Both analytical and numerical methods are presented. Owing to space limitations, some topics could not be covered. These include interfacial flows and the effects of viscosity, compressibility and surfactants. Some further developments of the theories described in the book can be found in the list of references.

Many results presented in the book have grown out of my research over the last 35 years and, of course, out of the research of the whole fluid mechanics community. References to the original papers are given. For this book, I have repeated the older numerical calculations with larger numbers of grid points than was possible at the time. I am pleased to report that the new results are in agreement with the earlier ones!

I am deeply indebted to my mentors, coworkers, students and friends who participated in the research. I feel very fortunate to have known them and I look forward to continuing these collaborations in the future. Special thanks are due to Scott Grandison for help with the figures, to David Tranah for his patience over the years in waiting for the manuscript, to Susan Parkinson for very careful copy-editing, to Caroline Brown for her help during production and to Cambridge University Press for publishing the book.