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978-0-521-81190-3 - Gravity-Capillary Free-Surface Flows
Jean-Marc Vanden-Broeck
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Introduction

Free-surface problems occur in many aspects of science and everyday life. They can be defined as problems whose mathematical formulation involves surfaces that have to be found as part of the solution. Such surfaces are called free surfaces. Examples of free-surface problems are waves on a beach, bubbles rising in a glass of champagne, melting ice, flows pouring from a container and sails blowing in the wind. In these examples the free surface is the surface of the sea, the interface between the gas and the champagne, the surface of the ice, the boundary of the pouring flow and the surface of the sail.

In this book we concentrate on applications arising in fluid mechanics. We hope to convince the reader of the beauty of such problems and to present the challenges faced when one attempts to describe these flows mathematically. Many of these challenges are resolved in the book but others are still open questions. We will always attempt to present fully nonlinear solutions without restricting assumptions on the smallness of some parameters. Our techniques are often numerical. However, it is the belief of the author that a deep understanding of the structure of the solutions cannot be gained by brute-force numerical approaches. It is crucial to combine numerical methods with analytical techniques, especially when singularities are present. Therefore analytical treatments will be presented whenever appropriate. We hope that the techniques discussed will be useful not only to researchers in the field but also to those working in areas other than fluid mechanics. For completeness the relevant concepts of fluid mechanics are reviewed in Chapter 2.

Free-surface flows fall into two main classes. The first is the class of such flows for which there are intersections between the free surface and a rigid surface. The classic example in this class is the flow due to a ship moving at the surface of a lake, which involves an intersection between the free surface (i.e. the surface of the lake) and a rigid surface (i.e. the hull of the ship). Other examples are jets leaving a nozzle, cavitating flows past an obstacle, bubbles attached to a wall and flows under a sluice gate. In each case there is a rigid surface (the nozzle, the obstacle, the wall or the gate) that intersects a free surface.

The second class contains free-surface flows for which there are no intersections between the free surface and a rigid wall. Here the classic example is the flow due to a submerged object moving below the surface of a lake. If the object is small compared with the size of the lake, it is then reasonable to regard the lake as being of infinite horizontal extent; then there is no intersection between the free surface and a rigid surface. Other examples include free bubbles rising in a fluid and solitary waves.

Chapter 3 is concerned with the theory of free-surface flows of the first class. We use the classical assumptions of potential flow theory (irrotational flows of inviscid and incompressible fluids) and proceed in stages of increasing complexity. In addition we restrict our attention to steady and two-dimensional flows (time-dependent and three-dimensional flows are considered in the last two chapters). In the first stage the effects of gravity and surface tension are neglected. Such free-surface flows are called free streamline flows. They are characterized by a constant velocity along the free surfaces. Conformal mapping techniques can then be used to find exact nonlinear solutions. This situation is fortunate since there are very few such solutions for free-surface flows. The main results of the free streamline theory are summarized in Section 3.1. The most important result for the remaining part of Chapter 3 is that the velocity and slope of the free surface must be continuous at a separation point (i.e. the intersection between a free surface and a rigid surface in two dimensions) but the curvature of the free surface is in general infinite. Since this curvature only enters the equations when surface tension is included in the dynamic boundary condition, we expect a gravity flow with 'small gravity' to be a regular perturbation of a free streamline flows, and a capillary flow with 'small surface tension' to be a singular perturbation. This is confirmed by the numerical results presented in Sections 3.2 and 3.3. It is shown in Section 3.2 that the presence of surface tension does not remove the infinite curvature at the separation points. On the contrary it makes the problem more singular by introducing a discontinuity in slope at the separation point. Depending on the angle between the free surface and the rigid boundary, the velocity is infinite or equal to zero at the separation point. The appearance of an infinite velocity is a limitation of the model. We show that a basic way to remove this singularity is to take into account the finite thickness of the rigid walls, i.e. to consider the walls as thin objects with a continuous slope. When surface tension is neglected, the free surfaces leave the wall tangentially but the position of the separation point along the walls is free. We then have a one-parameter family of solutions (the parameter defines the position of the separation point) and the question is to determine which value of the parameter is physically relevant. This is an example of a 'selection problem'. Selection problems are usually resolved by imposing an extra constraint on the problem or by including a previously neglected effect and taking the limit as this effect approaches zero. Here both approaches work. A unique position for the separation point is obtained by neglecting surface tension and imposing a constraint known as the Brillouin–Villat condition. Equivalently, the same position for the separation point is obtained by solving the problem with surface tension and then taking the limit as the surface tension approaches zero. We will show that the mechanism by which solutions with a small amount of surface tension are selected is related to the fact that, for each value of the surface tension, the separation point has only one position for which the free surface leaves the wall tangentially.

In Section 3.3 we turn our attention to the effects of gravity on free streamline solutions. We assume that gravity is acting vertically downwards and neglect surface tension (the combined effects of gravity and surface tension are covered in Section 3.4). We show that it is again possible for the free surface not to leave the walls tangentially, but the angle between the free surface and the wall must be such that the velocity is finite at the separation point (infinite velocities cannot occur on a free surface in the absence of surface tension). Local analysis shows that there are only three possible behaviours at the separation point. In the first there is a horizontal free surface at the separation point, in the second there is an angle of $120°$ between the free surface and the wall and in the third the free surface leaves the wall tangentially. We show by examples (e.g. flows emerging or pouring from containers) that these three possibilities occur in free-surface flows with gravity. The restriction to three local behaviours is to be contrasted with the cases including surface tension discussed in Section 3.2, where all angles between the walls and the free surfaces are in principle possible. This contrast suggests that some interesting behaviours might emerge if we combine the effects of gravity and surface tension; this is confirmed in Section 3.4. We show in this section that some free-surface flow problems possess a continuum of solutions when surface tension is neglected and an infinite discrete set of solutions when surface tension is taken into account. This discrete set reduces to a unique solution as the surface tension approaches zero. Therefore a small amount of surface tension can again be used to select solutions. One difference between this selection mechanism and that described in Section 3.2 is that there is a infinite discrete set of solutions when surface tension is included instead of one solution. Another difference is that the selection is associated with exponentially small terms in the surface tension. This implies that exponential asymptotics is required to predict the selected solutions analytically. As we shall see, exponential asymptotics plays an important role in many other free-surface flow problems such as the study of gravity–capillary waves (see Chapter 6) and free-surface flows generated by moving disturbances for small values of the Froude number or small values of the surface tension (see Chapter 8).

The results presented in Chapter 3 were obtained by a combination of various numerical schemes that the author has used successfully over the years to obtain highly accurate solutions for free-surface flow problems. They include series truncation techniques and boundary integral equation methods. The idea of the series truncation methods is to identify a rapidly convergent series representation for the solution that satisfies all the appropriate partial differential equations (for example the Laplace equation for potential flows) and all the linear boundary conditions. This often requires a local analysis to identify and remove the singularities associated with corners, stagnation points etc. The series is then truncated after a finite number of terms and the unknown coefficients are determined by satisfying the remaining nonlinear boundary condition (the pressure condition for free-surface flows) at appropriately chosen collocation points. This leads to a system of nonlinear algebraic equations which can be solved by iteration (for example by using Newton's method). Boundary integral equation methods are based on a reformulation of the problem as a system of nonlinear integro-differential equations for the unknown quantities on the free surface. These equations are then discretised and the resultant nonlinear algebraic equations solved by iteration. Such boundary integral equation methods have been used extensively by many researchers.

Insight into free-surface flows of the second class can be gained by studying the limitations of the classical linear theories. In particular we study in Chapter 4 the waves generated by a disturbance moving at a constant velocity (for example a submerged object or a pressure distribution). The results are qualitatively independent of the type of disturbance, and so most results in Chapter 4 are presented just for a pressure distribution with bounded support. A frame of reference moving with the pressure distribution is chosen and the flow is assumed to be steady. In the linear theory it is assumed that the disturbance is small enough for the flow to be a small perturbation of a uniform stream. The equations are then linearised (around a uniform stream) and the resulting linear equations solved by separation of variables and using Fourier transforms. These linear solutions can be expected to be a good approximation when the disturbance is small. In other words, if ϵ denotes the size of the disturbance, we expect the nonlinear solutions to approach the linear solutions as $\epsilon \to 0$. This is usually the case, but the problem is complicated by the fact that the solutions depend not only on ϵ but also on other parameters such as the Froude number

$$F = \frac{U}{(gH)^{1/2}}$$

and the capillary number

$$\alpha = \frac{Tg}{\rho U^4}.$$

Here U is the velocity of the disturbance, H the depth of the fluid, T the surface tension, g the acceleration of gravity and ρ the density of the fluid.

This leads to nonuniformities when these parameters approach critical values. These nonuniformities appear in an obvious way in the linear solutions. For example the linear theory for pure gravity flows (i.e. flows with $T = 0$) predicts infinite displacements of the free surface as $F \to 1$. This is unacceptable since the linear theory assumes small perturbations around a uniform stream and in particular small displacements of the free surface. More precisely, for any $F \neq 1$ the linear theory provides a good approximation of the nonlinear problem as $\epsilon \to 0$. However, for any ϵ, no matter how small, the linear solutions become invalid as $F \to 1$. A similar situation occurs for gravity–capillary flows. As α approaches a critical value α_H, the linear theory again predicts infinite displacements of the free surface. The critical value α_H depends on the depth H (for example $\alpha_H = 0.25$ in water of infinite depth).

The resolution of these nonuniformities requires a nonlinear theory. We develop in Chapter 7 such a theory by solving the fully nonlinear equations numerically. This approach has the advantage of not pre-assuming a particular type of expansion. Furthermore it gives solutions without any assumption on the size of ϵ. It also provides a valuable guide in deriving appropriate perturbation expansions for small or moderate values of ϵ. We show in Chapter 7 that the resolution of the nonuniformities is associated with solitary waves. Near the critical values of α and F, there are not only solutions that are perturbations of a uniform stream (the nonlinear equivalent of the linear solutions mentioned earlier) but also solutions that are perturbations of solitary waves. Some of these solitary waves are of the well-known Korteweg–de Vries type but others are solitary waves with decaying oscillatory tails.

As a preparation for the nonlinear results of Chapter 7 we present in Chapters 5 and 6 analytical and numerical solutions for nonlinear periodic and solitary waves. Such solutions describe the far-field behaviour of the nonlinear free-surface flows past disturbances described in Chapter 7. They are also interesting canonical free-surface flow problems. In particular we show that waves of the Korteweg–de Vries type have oscillatory tails of constant amplitude when surface tension is included. These waves are referred to as generalised solitary waves to distinguish them from true solitary waves, which are flat in the far field.

In Chapter 8 we consider further free-surface flows of the first class (i.e. flows for which the free surface intersects rigid walls). The solutions of Chapter 3 approach either an infinitely thin jet in the far field or are waveless. We study in Chapter 8 various extensions for which the free surface is characterised by a train of nonlinear waves in the far field. An attractive feature of some of these flows is that exact formulae can be derived for the amplitude of the waves in the far field. These relations provide analytical insight and can be used to check the accuracy of the numerical codes.

All the flows in Chapters 3–8 are assumed to be steady, two-dimensional and irrotational. The final three chapters of the book describe some extensions in which these assumptions are removed. In Chapter 9 we study solitary and periodic waves with constant vorticity. We show that there are new solution branches that do not have an equivalent for irrotational waves. In Chapter 10 we study some three-dimensional free-surface flows. In particular we calculate three-dimensional gravity–capillary solitary waves. These waves are characterised by decaying oscillations in the direction of propagation and monotonic decay in the direction perpendicular to the direction of propagation.

Chapter 11 is concerned with time-dependent free-surface flows. This is a very large subject involving problems such as breaking waves, stability, the breaking of jets etc. Here we limit our attention to the subject of gravity–capillary standing waves. This choice is motivated by the fact that these standing waves have properties similar to those of the travelling gravity–capillary waves presented in Chapter 6.

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Basic concepts

2.1 The equations of fluid mechanics

We start with a brief introduction to the equations of fluid mechanics. For further details see for example Batchelor [8] or Acheson [1].

All the fluids considered in this book are assumed to be inviscid and to have constant density ρ (i.e. to be incompressible).

Conservation of momentum yields the Euler equations

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p + \mathbf{X}, \tag{2.1}$$

where $\mathbf{u}$ is the vector velocity, p is the pressure and $\mathbf{X}$ is the body force. Here

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla \tag{2.2}$$

is the material derivative. We assume that the body force $\mathbf{X}$ derives from a potential Ω, i.e. that

$$\mathbf{X} = -\nabla\Omega. \tag{2.3}$$

In most applications considered in this book, the flow is assumed to be irrotational. Therefore

$$\nabla \times \mathbf{u} = 0. \tag{2.4}$$

Relation (2.4) implies that we can introduce a potential function ϕ such that

$$\mathbf{u} = \nabla\phi. \tag{2.5}$$

Conservation of mass gives

$$\nabla \cdot \mathbf{u} = 0. \tag{2.6}$$

Then (2.5) and (2.6) imply that ϕ satisfies Laplace's equation

$$\nabla^2\phi = 0. \tag{2.7}$$

Flows that satisfy (2.4)–(2.7) are referred to as potential flows. Using the identity

$$\mathbf{u}\cdot\nabla\mathbf{u} = \frac{1}{2}\nabla(\mathbf{u}\cdot\mathbf{u}) + (\nabla\times\mathbf{u})\times\mathbf{u}, \tag{2.8}$$

(2.4) and (2.2) yield

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial\mathbf{u}}{\partial t} + \frac{1}{2}\nabla(\mathbf{u}\cdot\mathbf{u}). \tag{2.9}$$

Substituting (2.9) into (2.1) and using (2.3) and (2.5) we obtain

$$\nabla\left(\frac{\partial\phi}{\partial t} + \frac{\mathbf{u}\cdot\mathbf{u}}{2} + \frac{p}{\rho} + \Omega\right) = 0. \tag{2.10}$$

After integration, (2.10) gives the well-known Bernoulli equation

$$\frac{\partial\phi}{\partial t} + \frac{\mathbf{u}\cdot\mathbf{u}}{2} + \frac{p}{\rho} + \Omega = F(t). \tag{2.11}$$

Here $F(t)$ is an arbitrary function of t. It can be absorbed in the definition of ϕ, and then (2.11) can be rewritten as

$$\frac{\partial\phi}{\partial t} + \frac{\mathbf{u}\cdot\mathbf{u}}{2} + \frac{p}{\rho} + \Omega = B, \tag{2.12}$$

where B is a constant. For steady flows (2.12) reduces to

$$\frac{\mathbf{u}\cdot\mathbf{u}}{2} + \frac{p}{\rho} + \Omega = B. \tag{2.13}$$

2.2 Free-surface flows

We introduce the concept of a free surface by contrasting the flow past a rigid sphere (see Figure 2.1) with that of the flow past a bubble (see Figure 2.2). Both flows are assumed to be steady and to approach a uniform stream with a constant velocity U as $x^2 + y^2 + z^2 \to \infty$; the effects of gravity are neglected. They can interpreted as the flows due to a rigid sphere or a bubble rising at a constant velocity U, when viewed in a frame of reference moving with the sphere or the bubble. The pressure p_b in the bubble is constant. We denote by S the surface of the sphere or bubble and by $\mathbf{n}$ the outward unit normal.

The flow past a sphere can be formulated as follows:

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \quad \text{outside} \quad S, \tag{2.14}$$

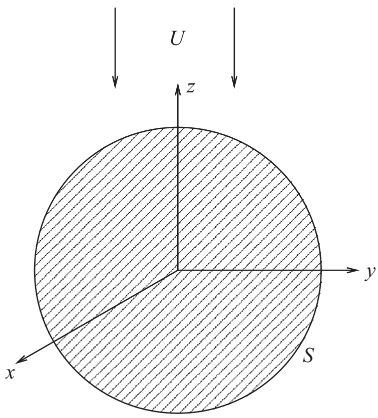

Fig. 2.1. The flow past a rigid sphere. The surface S of the sphere is described by $x^2 + y^2 + z^2 = R^2$, where R is the radius of the sphere.

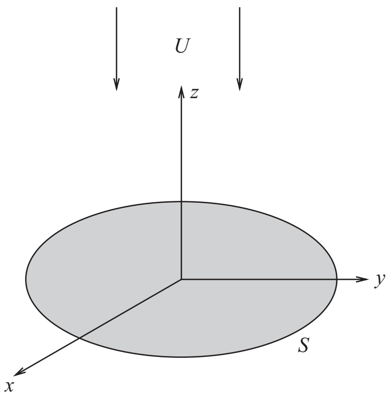

Fig. 2.2. The flow past a bubble. The surface S of the bubble is not known a priori and has to be found as part of the solution.

$$\frac{\partial \phi}{\partial n} = 0 \quad \text{on} \quad S \tag{2.15}$$

$$(\phi_x, \phi_y, \phi_z) \to (0, 0, -U) \quad \text{as} \quad x^2 + y^2 + z^2 \to \infty. \tag{2.16}$$

Equation (2.14) is Laplace's equation (2.7) expressed in cartesian coordinates. The boundary condition (2.15) is known as the kinematic boundary condition. It states that the normal component of the velocity vanishes on S.

Equations (2.14)–(2.16) form a linear boundary value problem whose solution is

$$\phi = -U\left[z + \frac{R^3 z}{2(x^2+y^2+z^2)^{3/2}}\right]. \tag{2.17}$$

Here R is the radius of the sphere.

We note that we have derived the solution (2.17) without using the Bernoulli equation (2.13), which for the present problem can be written as

$$\frac{1}{2}(\phi_x^2+\phi_y^2+\phi_z^2) + \frac{p}{\rho} = \frac{1}{2}U^2 + \frac{p_\infty}{\rho}. \tag{2.18}$$

Here p_∞ denotes the pressure as $x^2+y^2+z^2 \to \infty$. Equation (2.18) holds everywhere outside the sphere. In deriving (2.18) we have set $\Omega = 0$ in (2.13) and evaluated B by taking the limit $x^2+y^2+z^2 \to \infty$ in (2.13). Then, using (2.16) gives $B = U^2/2 + p_\infty/\rho$.

Equation (2.18) is nonlinear but it is only used if we want to calculate the pressure p inside the fluid. In other words the main problem is to find ϕ by solving the linear set of relations (2.14)–(2.16). We may then substitute the values (2.17) of ϕ into the nonlinear equation (2.18) if we wish to compute the pressure.

We now show that we need to use the nonlinear boundary condition (2.18) to solve for the potential ϕ for a flow past the bubble of Figure 2.2. This implies that, because of its nonlinearity, the flow past a bubble is a much harder problem to solve than the flow past a sphere. The potential function ϕ still satisfies (2.14)–(2.16). However, the main difference is that the shape of the surface S of the bubble is not known and has to be found as part of the solution. In other words the equation of the surface S is no longer given as it was for the flow past a sphere. Therefore we need an extra equation to find S. This equation uses (2.18) and can be derived as follows. First we relate the pressure p on the fluid side of S to the pressure $p_{\rm b}$ inside the bubble by using the concept of surface tension. If we draw a line on a fluid surface (such as S), the fluid on the right of the line is found to exert a tension T, per unit length of the line, on the fluid to the left. We call T the surface tension coefficient. It depends on the fluid and also on the temperature. It can be shown (see for example Batchelor [8]) that

$$p - p_{\rm b} = TK = T\left(\frac{1}{R_1} + \frac{1}{R_2}\right). \tag{2.19}$$

Here R_1 and R_2 are the principal radii of curvature of the fluid surface: they are counted positive when the centres of curvature lie inside the fluid. The