London Mathematical Society Lecture Note Series. 294

Introduction to Operator Space Theory

Gilles Pisier
Texas A&M University & University of Paris 6

CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge, CB2 2RU, UK
40 West 20th Street, New York, NY 10011-4211, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa

http://www.cambridge.org

First published 2003

Printed in the United Kingdom at the University Press, Cambridge

Typeface CMR 10/12 pt. *System* LaTeX 2_ε [TB]

A catalog record for this book is available from the British Library.

Library of Congress Cataloging in Publication Data

Pisier, Gilles, 1950–
Introduction to operator space theory / Gilles Pisier.
p. cm. – (London Mathematical Society lecture note series ; 294)
Includes bibliographical references and index.
ISBN 0-521-81165-1 (pbk.)
1. Operator spaces. I. Title. II. Series.
QA322.2 .P545 2003
515′.732–dc21 2002031358

ISBN 0 521 81165 1 paperback

CONTENTS

0. Introduction 1

Part I. INTRODUCTION TO OPERATOR SPACES

1. Completely bounded maps 17
2. The minimal tensor product. Ruan's theorem. Basic operations 28
   - 2.1. Minimal tensor product 28
   - 2.2. Ruan's theorem 34
   - 2.3. Dual space 40
   - 2.4. Quotient space 42
     - Quotient by a subspace 42
     - Quotient by an ideal 43
   - 2.5. Bidual. Von Neumann algebras 47
   - 2.6. Direct sum 51
   - 2.7. Intersection, sum, complex interpolation 52
   - 2.8. Ultraproduct 59
   - 2.9. Complex conjugate 63
   - 2.10. Opposite 64
   - 2.11. Ruan's theorem and quantization 65
   - 2.12. Universal objects 67
   - 2.13. Perturbation lemmas 68
3. Minimal and maximal operator space structures 71
4. Projective tensor product 81
5. The Haagerup tensor product 86
   - Basic properties 86
   - Multilinear factorization 92
   - Injectivity/projectivity 93
   - Self-duality 93
   - Free products 98
   - Factorization through R or C 101
   - Symmetrized Haagerup tensor product 102
   - Complex interpolation 106
6. Characterizations of operator algebras 109
7. The operator Hilbert space 122
   - Hilbertian operator spaces 122
   - Existence and unicity of OH. Basic properties 122
   - Finite-dimensional estimates 130
   - Complex interpolation 135
   - Vector-valued L_p-spaces, either commutative or noncommutative 138
8. Group C^*-algebras. Universal algebras and unitization for an operator space 148
9. Examples and comments 165
   - 9.1. A concrete quotient: Hankel matrices 165

9.2. Homogeneous operator spaces 172
9.3. Fermions. Antisymmetric Fock space. Spin systems 173
9.4. The Cuntz algebra O_n 175
9.5. The operator space structure of the classical L_p-spaces 178
9.6. The C^*-algebra of the free group with n generators 182
9.7. Reduced C^*-algebra of the free group with n generators 183
9.8. Operator space generated in the usual L_p-space by Gaussain random variables or by the Rademacher functions 191
9.9. Semi-circular systems in Voiculescu's sense 200
9.10. Embeddings of von Neumann algebras into ultraproducts 210
9.11. Dvoretzky's theorem 215
10. Comparisons 217

Part II. OPERATOR SPACES AND C^*-TENSOR PRODUCTS
11. C^*-norms on tensor products. Decomposable maps. Nuclearity 227
12. Nuclearity and approximation properties 240
13. $C^*(\mathbb{F}_\infty) \otimes B(H)$ 252
14. Kirchberg's theorem on decomposable maps 261
15. The Weak Expectation Property (WEP) 267
16. The Local Lifting Property (LLP) 275
17. Exactness 285
18. Local reflexivity 303
Basic properties 303
A conjecture on local reflexivity and OLLP 305
Properties C, C', and C''. Exactness versus local reflexivity 309
19. Grothendieck's theorem for operator spaces 316
20. Estimating the norms of sums of unitaries: Ramanujan graphs, property T, random matrices 324
21. Local theory of operator spaces. Nonseparability of OS_n 334
22. $B(H) \otimes B(H)$ 348
23. Completely isomorphic C^*-algebras 354
24. Injective and projective operator spaces 356

Part III. OPERATOR SPACES AND NON-SELF-ADJOINT OPERATOR ALGEBRAS
25. Maximal tensor products and free products of operator algebras 365
26. The Blecher-Paulsen factorization. Infinite Haagerup tensor products 384
27. Similarity problems 396
28. The Sz.-Nagy-Halmos similarity problem 407

Solutions to the exercises 418

References 457
Subject index 477
Notation index 479

Chapter 1. Completely Bounded Maps

Let us start by recalling the definition and a few facts on C^*-algebras:

Definition 1.1. *A C^*-algebra is a Banach $*$-algebra satisfying the identity*

$$\|x^*x\| = \|x\|^2$$

for any element x in the algebra.

The simplest example is the space

$$B(H)$$

of all bounded operators on a Hilbert space H, equipped with the operator norm. More generally, any closed subspace

$$A \subset B(H)$$

stable under product and involution is a C^*-algebra.

By classical results (Gelfand and Naimark) we know that every C^*-algebra can be realized as a closed self-adjoint subalgebra of $B(H)$. Moreover, we also know that every commutative unital C^*-algebra can be identified with the space $C(T)$ of all complex-valued continuous functions $f\colon T \to \mathbb{C}$ on some compact space T. If A has no unit, A can be identified with the space $C_0(T)$ of all complex-valued continuous functions, vanishing at infinity, on some locally compact space T.

Of course the object of C^*-algebra theory (as developed in the last 50 years; cf. [KaR, Ta3]) is the classification of C^*-algebras. Similarly, the object of Banach space theory is the classification of Banach spaces.

In the last 25 years, it is their classification *up to isomorphism* (and *NOT* up to isometry) that has largely predominated (cf., e.g., [LT1–3, P4]).

This already indicates one major difference between these two fields since, if A_1 and A_2 are two C^*-algebras,

$$A_1 \text{ isomorphic to } A_2 \Rightarrow A_1 \text{ isometric to } A_2.$$

In particular, a C^*-algebra admits a *unique* C^*-norm. So there is no "isomorphic theory" of C^*-algebras. However, in recent years, operator algebraists have found the need to relax the structure of C^*-algebras and consider more general objects called *operator systems*. These are subspaces of $B(H)$ containing the unit that are stable under the involution but not under the product. The theory of operator systems was developed using the order structure repeatedly, and it is still mostly an isometric theory. The natural morphisms here are the "completely positive" maps (cf. [St, Ar1]). We refer the reader to a survey by Effros [E1] and a series of papers by Choi and Effros (especially [CE3]). Even more recently, operator algebraists have done a radical simplification and considered just "operator spaces":

Definition 1.2. *An operator space is a closed subspace of $B(H)$.*

Equivalently, since we can think of C^*-algebras as closed self-adjoint subalgebras of $B(H)$, we can think of operator spaces as closed subspaces of C^*-algebras.

Operator space theory can be considered as a merger of C^*-algebra theory and Banach space theory.

It is important to immediately observe that any Banach space can appear as a closed subspace of a C^*-algebra. Indeed, for any Banach space X (with the dual unit ball denoted by B_{X^*}), if we let

$$T = (B_{X^*}, \sigma(X^*, X)),$$

then T is compact and we have an isometric embedding

$$X \subset C(T).$$

Hence, since $C(T)$ is a C^*-algebra (and $C(T) \subset B(H)$ with $H = \ell_2(T)$), X also appears among operator spaces. So operator spaces are just ordinary Banach spaces X but equipped with an extra structure in the form of an embedding

$$X \subset B(H).$$

The main difference between the category of Banach spaces and that of operator spaces lies not in the spaces but in the *morphisms*. We need morphisms that somehow keep track of the extra information contained in the data of the embedding $X \subset B(H)$; the maps that do just this are the *completely bounded maps.*

Definition 1.3. *Let $E \subset B(H)$ and $F \subset B(K)$ be operator spaces and consider a map*

$$\begin{array}{ccc} B(H) & & B(K) \\ \cup & & \cup \\ E & \xrightarrow{u} & F \end{array}$$

For any $n \geq 1$, let

$$M_n(E) = \{(x_{ij})_{ij\leq n} \mid x_{ij} \in E\}$$

be the space of $n \times n$ matrices with entries in E. In particular, we have a natural identification

$$M_n(B(H)) \simeq B(\ell_2^n(H)),$$

where $\ell_2^n(H)$ means $\underbrace{H \oplus H \oplus \cdots \oplus H}_{n \text{ times}}$. Thus, we may equip $M_n(B(H))$ and a fortiori its subspace

$$M_n(E) \subset M_n(B(H))$$

with the norm induced by

$$B(\ell_2^n(H)).$$

Then, for any $n \geq 1$, the linear map $u\colon E \to F$ allows us to define a linear map

$$u_n\colon M_n(E) \longrightarrow M_n(F)$$

defined by

$$u_n \begin{pmatrix} & \vdots & \\ \dots & x_{ij} & \dots \\ & \vdots & \end{pmatrix} = \begin{pmatrix} & \vdots & \\ \dots & u(x_{ij}) & \dots \\ & \vdots & \end{pmatrix}.$$

A map $u\colon E \to F$ is called completely bounded (in short c.b.) if

$$\sup_{n\geq 1} \|u_n\|_{M_n(E)\to M_n(F)} < \infty.$$

We define

$$\|u\|_{cb} = \sup_{n\geq 1} \|u_n\|_{M_n(E)\to M_n(F)},$$

and we denote by

$$CB(E,F)$$

the Banach space of all c.b. maps from E into F equipped with the c.b. norm.

This space will replace the space $B(E,F)$ of all bounded operators from E into F. (We will see later on that it can be equipped with an operator space structure.)

If $G \subset B(L)$ is another operator space and if $v\colon F \to G$ is c.b., then the compositon $vu\colon E \to G$ clearly remains c.b. and we have

$$\|vu\|_{cb} \leq \|v\|_{cb}\|u\|_{cb}.$$

Of course, when $n = 1$, 1×1 matrices are just elements of E, so $u_1\colon M_1(E) \to M_1(F)$ is nothing but u itself. In particular we have

$$\|u\| \leq \|u\|_{cb}$$

and

$$CB(E,F) \subset B(E,F).$$

When $\|u\|_{cb} \leq 1$, we say that u is "completely contractive" (or "a complete contraction").

The notion of isometry is replaced by that of "complete isometry": A map $u\colon E \to F$ is called a complete isometry ($= u$ is completely isometric) if

$$u_n\colon M_n(E) \to M_n(F)$$

is an isometry for all $n \geq 1$.

Similarly, a map $u\colon E \to F$ is called completely positive (in short c.p.) if $u_n\colon M_n(E) \to M_n(F)$ is positive for all n (in the order structure induced by the C^*-algebras $M_n(B(H))$ and $M_n(B(K))$. Moreover, we should emphasize

Definition 1.4. *Two operator spaces E, F are called completely isomorphic if there is a linear isomorphism $u\colon E \to F$ such that u and u^{-1} are c.b.*

We will say that E, F are completely isometric if there is a linear isomorphism $u\colon E \to F$ that is a complete isometry (or, equivalently, that satisfies $\|u\|_{cb} = \|u^{-1}\|_{cb} = 1$). In that case, we will often identify these spaces, although this might sometimes be abusive.

Proposition 1.5. *Let $A_1 \subset B(H_1)$, $A_2 \subset B(H_2)$ be two C^*-algebras; let $E_1 \subset A_1$, $E_2 \subset A_2$ be two operator spaces; let $\pi\colon A_1 \to A_2$ be a representation such that $\pi(E_1) \subset E_2$; and let $u\colon E_1 \to E_2$ be the restriction of π. Then u is completely bounded and $\|u\|_{cb} \leq 1$. Moreover, if π is injective, u is completely isometric.*

Proof. It is well known that a C^*-algebra representation π automatically has norm at most 1 and a closed range (cf. [Ta3, p. 21-22]). Therefore, $\|\pi\| \leq 1$, but since $\pi_n\colon M_n(A_1) \to M_n(A_2)$ also is a C^*-algebra representation, we again have $\|\pi_n\| \leq 1$ for all n, and hence $\|u\|_{cb} \leq 1$. Moreover, if a representation π is injective, it is necessarily isometric (since its inverse must also have norm at most 1), and hence π_n itself is isometric for all n. ■

We can measure the "c.b. distance" between E and F by setting

$$d_{cb}(E, F) = \inf\{\|u\|_{cb}\|u^{-1}\|_{cb} \mid u\colon E \to F \text{ complete isomorphism}\}.$$

If E, F are not completely isomorphic, we will set

$$d_{cb}(E, F) = \infty.$$

Examples. When E, F are Banach spaces we can view them as operator spaces via the embeddings

$$E \subset C(B_{E^*}), \qquad F \subset C(B_{F^*}).$$

This is of course not a very interesting operator space structure, but it shows that – to some extent – Banach space theory can be viewed as embedded into operator space theory, since for a map

$$\begin{array}{ccc} C(B_{E^*}) & & C(B_{F^*}) \\ \cup & & \cup \\ E & \xrightarrow{u} & F \end{array}$$

we have necessarily

$$u \text{ bounded } \Leftrightarrow u \quad \text{c.b.}$$

and

$$\|u\| = \|u\|_{cb}.$$

Actually (see Proposition 1.10), this remains true when E is an arbitrary operator space, assuming only that F is equipped with its "commutative structure" as above. Moreover, it is easy to check that $\|u\| = \|u\|_{cb}$ for any rank one mapping u between operator spaces. This implies of course that if $\dim(E) = 1$, then its commutative operator space structure is the only possible one on E.

Here are more interesting examples:

In $B(\ell_2)$ consider the column Hilbert space

$$C = \overline{\text{span}}\{e_{i1} \mid i \in \mathbb{N}\} \tag{1.1}$$

and the row Hilbert space

$$R = \overline{\text{span}}\{e_{1j} \mid j \in \mathbb{N}\}. \tag{1.2}$$

We will also need their finite-dimensional versions:

$$\begin{aligned} C_n &= \text{span}\{e_{i1} \mid 1 \le i \le n\} \\ R_n &= \text{span}\{e_{1j} \mid 1 \le j \le n\}. \end{aligned}$$

Then, as Banach spaces, R and C are indistinguishable, since they are both isometric to ℓ_2, that is, we have

$$\forall x \in \ell_2 \qquad \left\|\sum x_i e_{i1}\right\|_{B(\ell_2)} = \left(\sum |x_i|^2\right)^{1/2} = \left\|\sum x_j e_{1j}\right\|_{B(\ell_2)}. \tag{1.3}$$

However, as operator spaces, they are not isomorphic. Actually they are extremely far apart, since we have (see [Mat1–2])

$$d_{cb}(R_n, C_n) = n, \tag{1.4}$$

which is the maximal distance possible between any two n-dimensional operator spaces. Actually, it can be shown (cf., e.g., [P5, p. 270], [ER4]) that for any

$$u\colon\ R \to C \qquad (\text{or } u\colon\ C \to R)$$

we have (HS stands for Hilbert-Schmidt)

$$\|u\|_{cb} = \|u\|_{HS}. \tag{1.5}$$

For the proof, see the solution to Exercise 1.1. It follows that, for any isomorphism $u\colon\ R_n \to C_n$, we have

$$n = tr(u^{-1}u) \le \|u\|_{HS}\|u^{-1}\|_{HS} = \|u\|_{cb}\|u^{-1}\|_{cb},$$

which implies $d_{cb}(R_n, C_n) \geq n$. For the converse it suffices to observe that the map u: $R_n \to C_n$ taking e_{1j} to e_{j1} (=transposition) satisfies $\|u\|_{cb} = \|u\|_{HS} = \sqrt{n}$ and $\|u^{-1}\|_{cb} = \|u^{-1}\|_{HS} = \sqrt{n}$.

Letting $n \to \infty$, this gives us a simple example of an isometric map from R to C that is not c.b. A fortiori, the transposition $x \to {}^t x$ is isometric but is not c.b. either on $B(\ell_2)$ or on $\mathcal{K}$. More precisely, let τ_n: $M_n \to M_n$ denote the transposition of matrices. Then one can prove (see Exercise 1.2)

$$\|\tau_n\|_{cb} = n. \tag{1.6}$$

These examples R and C are fundamental. Indeed, using the Haagerup tensor product (denoted by $\otimes_h$) presented in Chapter 5, one can reconstruct the whole of $B(\ell_2)$ or $B(H)$ using R and C as the basic "building blocks" more precisely, we have $M_n = C_n \otimes_h R_n$, $K(\ell_2) = C \otimes_h R$, and of course $B(\ell_2) = K(\ell_2)^{**}$.

More generally, let H_1, H_2 be two Hilbert spaces and let $\mathcal{H} = H_1 \oplus H_2$. The mapping

$$x \to \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix}$$

is an isometric embedding of $B(H_1, H_2)$ into $B(\mathcal{H})$. Using this, we can view $B(H_1, H_2)$ as an operator space. Note that the norm induced on $M_n(B(H_1, H_2))$ by $M_n(B(\mathcal{H}))$ coincides with the norm of the space $B(\ell_2^n(H_1), \ell_2^n(H_2))$. In particular, we will often use the following:

Notation. Let H be an arbitrary Hilbert space. For any $h \in H$, we denote by $h_c \in B(\mathbb{C}, H)$ and $h_r \in B(H^*, \mathbb{C})$ the isometric embeddings defined by

$$\forall \lambda \in \mathbb{C} \qquad h_c(\lambda) = \lambda h$$
$$\forall \xi \in H^* \qquad h_r(\xi) = \langle \xi, h \rangle.$$

We will denote by H_c and H_r the resulting operator space structures on H. Recall that the dual H^* can be *canonically* identified with the complex conjugate Hilbert space $\overline{H}$.

In particular, we have

$$C = (\ell_2)_c \quad \text{and} \quad R = (\ell_2)_r.$$

Let a: $H_1 \to H_1$ and b: $H_2 \to H_2$ be bounded operators and let u_{ab}: $B(H_1, H_2) \to B(H_1, H_2)$ be defined by $u_{ab}(T) = bTa$. Clearly, u_{ab} is c.b. and $\|u_{ab}\|_{cb} \leq \|a\|\|b\|$. Taking either H_1 or H_2 one-dimensional, this implies immediately for any Hilbert space H

$$\forall u\colon\ H_c \to H_c \quad \|u\|_{cb} = \|u\| \quad \text{and} \quad \forall v\colon\ H_r \to H_r \quad \|v\|_{cb} = \|v\|.$$

Indeed, we have $u(h_c) = [u(h)]_c$ and analogously for r. In particular,

$$\forall u\colon C \to C \quad \|u\|_{cb} = \|u\| \quad \text{and} \quad \forall v\colon R \to R \quad \|v\|_{cb} = \|v\|. \tag{1.7}$$

The theory of c.b. maps clearly is the basis for operator space theory. It emerged in the early 1980s through the works of Wittstock [Wit1–2], Haagerup [H3], and Paulsen [Pa3], who proved (independently) a fundamental factorization and extension theorem for c.b. maps. This factorization is a generalization of earlier important work by Stinespring and Arveson ([St, Ar1]) who proved a factorization/extension theorem for completely positive maps.

Theorem 1.6. *(Fundamental Factorization/Extension Theorem.) Consider a c.b. map*

$$\begin{array}{ccc} B(H) & & B(K) \\ \cup & & \cup \\ E & \xrightarrow{u} & F \end{array}$$

Then there is a Hilbert space $\widehat{H}$, a representation

$$\pi\colon\ B(H) \longrightarrow B(\widehat{H}),$$

and operators $V_1\colon\ K \to \widehat{H}$, $V_2\colon\ \widehat{H} \to K$ such that $\|V_1\|\ \|V_2\| = \|u\|_{cb}$ and

$$\forall x \in E \qquad u(x) = V_2\pi(x)V_1. \tag{1.8}$$

Conversely, if (1.8) *holds then u is c.b. and $\|u\|_{cb} \le \|V_1\|\ \|V_2\|$ (in addition, if $V_1 = V_2^*$, then u is completely positive). Moreover, u admits a c.b. extension $\widetilde{u}\colon\ B(H) \to B(K)$*

$$\begin{array}{ccc} B(H) & \xrightarrow{\widetilde{u}} & B(K) \\ \cup & & \cup \\ E & \xrightarrow{u} & F \end{array}$$

such that $\|\widetilde{u}\|_{cb} = \|u\|_{cb}$.

For a proof, see either [Pa1], [P10], or [P5]; the latter extends to the case when H and K are Banach spaces.

This theorem explains the claim that c.b. maps keep track of the operator space structure. Indeed, it shows that (as explained in the Introduction) every c.b. map is the restriction of the composition of a representation and a two-sided multiplication.

For emphasis and for later reference, we state as separate corollaries parts of Theorem 1.6 that will be used frequently in the sequel. The first is the extension property of $B(K)$, which can be viewed as an operator-valued version of the Hahn-Banach Theorem:

Corollary 1.7. *Let $E, \widetilde{E}$ be operator spaces so that $E \subset \widetilde{E} \subset B(H)$. Then any c.b. map $u\colon E \to B(K)$ admits a c.b. extension $\widetilde{u}\colon \widetilde{E} \to B(K)$ with $\|\widetilde{u}\|_{cb} = \|u\|_{cb}$.*

Proof. We simply let $\widetilde{u}$ be the restriction of $x \mapsto V_2\pi(x)V_1$ to $\widetilde{E}$. ■

The second is the dilation property of unital complete contractions:

Corollary 1.8. *Let $E \subset B(H)$ be an operator space containing I. Consider a map $u\colon E \to B(K)$. If $u(I) = I$ and $\|u\|_{cb} = 1$, then there is a Hilbert space $\widehat{H}$ with $K \subset \widehat{H}$ and a representation $\pi\colon B(H) \to B(\widehat{H})$ such that*

$$\forall\, x \in E \qquad u(x) = P_K \pi(x)_{|K}.$$

In particular, u is completely positive.

Proof. By Theorem 1.6, we have $u(\cdot) = V_2\pi(\cdot)V_1$. By homogeneity, we may assume $\|V_1\| = \|V_2\| = 1$. Since $I = u(I) = V_2\pi(I)V_1 = V_2V_1$, V_1 must be an isometric embedding of K into $\widehat{H}$. Identifying K with $V_1(K)$, $u(\cdot) = V_2\pi(\cdot)V_1$ becomes $u(\cdot) = P_K\pi(\cdot)_{|K}$. ■

Finally, the third corollary is the decomposability of c.b. maps as linear combinations of c.p. maps:

Corollary 1.9. *Any c.b. map $u\colon E \to B(K)$ can be decomposed as $u = u_1 - u_2 + i(u_3 - u_4)$, where u_1, u_2, u_3, u_4 are c.p. maps with $\|u_j\|_{cb} \le \|u\|_{cb}$.*

Proof. By Theorem 1.6, we have $u(\cdot) = V_2\pi(\cdot)V_1$. Let us denote $V = V_1$ and $V_2 = W^*$, so that $u(\cdot) = W^*\pi(\cdot)V$. Then the result simply follows from the polarization formula: We define u_1, u_2, u_3, u_4 by

$$u_1(\cdot) = 4^{-1}(V+W)^*\pi(\cdot)(V+W), \qquad u_2(\cdot) = 4^{-1}(V-W)^*\pi(\cdot)(V-W),$$
$$u_3(\cdot) = 4^{-1}(V+iW)^*\pi(\cdot)(V+iW), \quad u_4(\cdot) = 4^{-1}(V-iW)^*\pi(\cdot)(V-iW).$$

Then $\|u_j\|_{cb} \le 1$ for $j = 1,2,3,4$ and $u = u_1 - u_2 + i(u_3 - u_4)$. (Note that actually $\|u_1 + u_2\|_{cb} \le 1$ and $\|u_3 + u_4\|_{cb} \le 1$). ■

Proposition 1.10. *Let $F \subset B(H)$ be an operator space. Let A_F be the C^*-algebra generated by F.*

(i) For any $n \ge 1$ and any x in $M_n(F)$ we have

$$\|x\|_{M_n(F)} \ge \sup\left\{ \left\|\sum \lambda_i \mu_j x_{ij}\right\|_F \mid \lambda_i \in \mathbb{C}, \mu_j \in \mathbb{C}, \sum |\lambda_i|^2 \le 1, \sum |\mu_j|^2 \le 1 \right\}.$$

(ii) Assume either A_F commutative or $\dim(F) = 1$. *Then we have equality in (i). Moreover, in either case, if E is an arbitrary operator space, any bounded map $u\colon E \to F$ is c.b. and satisfies $\|u\|_{cb} = \|u\|$.*

(iii) For any E, F, every finite-rank map $u\colon E \to F$ is c.b.

Proof. (i) is an easy exercise. When A_F is commutative, we can assume $A_F = C_0(\Omega)$ and also $M_n(A_F) = C_0(\Omega; M_n)$ for some locally compact space Ω. Then equality in (i) is very simple to check. When $\dim(F) = 1$, the verification is again an easy exercise. The second assertion in (ii) then follows by applying (i) in E and the equality case in F. Thus any map of rank one is c.b., which implies the same for any finite-rank map. ■

Note that (ii) implies that (not too surprisingly!) there is only one abstract operator space structure on $\mathbb{C}$.

Remark 1.11. Let E_1, E_2 be two Banach spaces. Consider an element $x = \sum a_i \otimes b_i$ in the algebraic tensor product $E_1 \otimes E_2$. The "injective" tensor norm (in Grothendieck's sense) is defined as

$$\begin{aligned}\|x\|_\vee &= \sup\{|\langle \xi_1 \otimes \xi_2, x\rangle| \mid \xi_1 \in B_{E_1^*}, \xi_2 \in B_{E_2^*}\}\\ &= \sup\left\{\left|\sum \xi_1(a_i)\xi_2(b_i)\right| \;\middle|\; \xi_1 \in B_{E_1^*}, \xi_2 \in B_{E_2^*}\right\}.\end{aligned}$$

Note that we can write alternatively

$$\|x\|_\vee = \sup_{\xi_1 \in B_{E_1^*}} \left\{\left\|\sum \xi_1(a_i) b_i\right\|_{E_2}\right\} = \sup_{\xi_2 \in B_{E_2^*}} \left\{\left\|\sum a_i \xi_2(b_i)\right\|_{E_1}\right\}.$$

We denote by $E_1 \overset{\vee}{\otimes} E_2$ the completion of $E_1 \otimes E_2$ for this norm, and we call it the injective tensor product of E_1, E_2.

In particular, for any Banach space E, we have for any $x \in M_n \otimes E$

$$\begin{aligned}\|x\|_{M_n \overset{\vee}{\otimes} E} &= \sup\left\{\left\|\sum \lambda_i \mu_j x_{ij}\right\|_E \;\middle|\; (\lambda_i), (\mu_j) \in \mathbb{C}^n, \sum |\lambda_i|^2 \le 1, \sum |\mu_j|^2 \le 1\right\}\\ &= \sup\left\{\left\|\sum e_{ij}\xi(x_{ij})\right\|_{M_n} \;\middle|\; \xi \in B_{E^*}\right\}.\end{aligned} \tag{1.9}$$

Note that for any locally compact space Ω and any Banach space B (in particular for $B = M_n$) we have an isometric isomorphism

$$C_0(\Omega, B) = C_0(\Omega) \overset{\vee}{\otimes} B.$$

Remark. Let $\alpha(n)$ be the best constant C such that, for any E, F, any map $u\colon E \to F$ of rank n satisfies

$$\|u\|_{cb} \le C\|u\|.$$

We will see in Theorem 3.8 later that $n/2 \le \alpha(n) \le n$ and in Chapter 7 that $\alpha(n) \le n/2^{1/4}$ (due to Eric Ricard), but the exact value of $\alpha(n)$ does not seem to be known.

The following result due to R. Smith [Sm2] is often useful.

Proposition 1.12. *Consider $E \subset B(H)$ and u: $E \to M_N = B(\ell_2^N, \ell_2^N)$. Then we have*

$$\|u\|_{cb} = \|u_N\|_{M_N(E) \to M_N(M_N)}.$$

Proof. This can be proved using the fact that, if $x_1, \ldots, x_n$ is a finite subset of ℓ_2^N with $\sum_1^n \|x_i\|^2 \leq 1$, then (we leave this as an exercise for the reader) there are an $n \times N$ scalar matrix $b = (b_{jk})$ with $\|(b_{jk})\| \leq 1$ and vectors $\widetilde{x}_1, \ldots, \widetilde{x}_N$ in ℓ_2^N such that $\sum_1^N \|\widetilde{x}_i\|^2 \leq 1$ and

$$\forall j \leq n \quad x_j = \sum_{k=1}^{N} b_{jk} \widetilde{x}_k.$$

Similarly, for any $y_1, \ldots, y_n$ in ℓ_2^N there are a scalar matrix $c = (c_{il})$ with $\|(c_{il})\| \leq 1$ and $\widetilde{y}_1, \ldots, \widetilde{y}_N$ in ℓ_2^N such that $\sum_1^N \|\widetilde{y}_i\|^2 \leq 1$ and

$$\forall i \leq n \quad y_i = \sum_{l=1}^{N} c_{il} \widetilde{y}_l.$$

Hence for any $n \times n$ marix (a_{ij}) in $M_n(E)$ we have

$$\sum_{i,j=1}^{n} \langle u(a_{ij}) x_j, y_i \rangle = \sum_{k,l=1}^{N} \langle u(\alpha_{lk}) \widetilde{x}_k, \widetilde{y}_l \rangle,$$

where $(\alpha_{lk}) \in M_N(E)$ is defined by $(\alpha_{lk}) = c^*.(a_{ij}).b$ (matrix product). Therefore:

$$\|(u(a_{ij}))\|_{M_n(M_N)} \leq \|(u(\alpha_{kl}))\|_{M_N(M_N)} \leq \|u_N\|_{M_N(E) \to M_N(M_N)} \|(\alpha_{lk})\|_{M_N(E)}$$

$$\leq \|u_N\|_{M_N(E) \to M_N(M_N)} \|(a_{ij})\|_{M_N(E)}.$$ ■

Remark 1.13. Consider $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ in $B(H)$. Let $a \in M_n(B(H))$ (resp. $b \in M_n(B(H))$) be the $n \times n$ matrix that has $a_1, \ldots, a_n$ (resp. $b_1, \ldots, b_n$) on its first column (resp. row) and zero elsewhere; that is, we have

$$a = \begin{pmatrix} a_1 & \\ \vdots & \\ \vdots & \bigcirc \\ \vdots & \\ a_n & \end{pmatrix} \qquad b = \begin{pmatrix} b_1 & \ldots & b_n \\ & \bigcirc & \end{pmatrix}.$$

Then

$$\|a\| = \left\| \sum a_i^* a_i \right\|_{B(H)}^{1/2} \quad \text{and} \quad \|b\| = \left\| \sum b_i b_i^* \right\|_{B(H)}^{1/2}. \tag{1.10}$$

Indeed, we have $\|a\| = \|a^*a\|^{1/2}$ and $\|b\| = \|bb^*\|^{1/2}$. Moreover, we have $\|ba\| \leq \|b\|\|a\|$, and hence

$$\|\sum b_i a_i\| \leq \left\|\sum b_i b_i^*\right\|_{B(H)}^{1/2} \left\|\sum a_i^* a_i\right\|_{B(H)}^{1/2}. \tag{1.11}$$

More generally, for any $x = (x_{ij})$ in $M_n(B(H))$ we have $\|bxa\| \leq \|b\|\|x\|\|a\|$, and hence

$$\|\sum_{i,j} b_i x_{ij} a_j\| \leq \left\|\sum b_i b_i^*\right\|_{B(H)}^{1/2} \|x\|_{M_n(B(H))} \left\|\sum a_j^* a_j\right\|_{B(H)}^{1/2}. \tag{1.12}$$

Note that it is easy to extend this remark to $n = \infty$.

Exercises

Exercise 1.1. Prove (1.5).

Exercise 1.2. Prove (1.6).

Exercise 1.3. Let $u\colon E \to F$ be a mapping between operator spaces. Show that for any $a_1, \ldots, a_n$ in E we have

$$\|\sum u(a_j)^* u(a_j)\|^{1/2} \leq \|u\|_{cb} \|\sum a_j^* a_j\|^{1/2} \text{ and } \|\sum u(a_j) u(a_j)^*\|^{1/2} \leq \|u\|_{cb} \|\sum a_j a_j^*\|^{1/2}.$$

Exercise 1.4. Let $u\colon E \to F$ be a mapping between operator spaces. Show that

$$\|u\|_{cb} = \sup_{n \geq 1} \{\|vu\|_{cb} \mid v\colon F \to M_n \quad \|v\|_{cb} \leq 1\}.$$

Exercise 1.5. (Schur Multipliers) (i) Let $\{x_i \mid i \leq n\}$ and $\{y_j \mid j \leq n\}$ be elements in the unit ball of a Hilbert space K. Then the mapping $u\colon M_n \to M_n$ defined by $u([a_{ij}]) = [a_{ij} \langle x_i, y_j \rangle]$ is a complete contraction. In addition, if $x_i = y_i$ for all i, then u is completely positive.

(ii) More generally, let S, T be arbitrary sets. We will identify an element of $B(\ell_2(T), \ell_2(S))$ with a matrix $\{a(s,t) \mid (s,t) \in S \times T\}$ in the usual way.

Let $\{x_s \mid s \in S\}$ and $\{y_t \mid t \in T\}$ be elements in the unit ball of a Hilbert space K. Then the mapping $u\colon B(\ell_2(T), \ell_2(S)) \to B(\ell_2(T), \ell_2(S))$ that takes $(a(s,t))_{(s,t)\in S\times T}$ to $(a(s,t)\langle x_s, y_t\rangle)_{(s,t)\in S\times T}$ is a complete contraction. In addition, if $S = T$ and $x_t = y_t$ for all t, then u is completely positive.