Chapter 0. Introduction

The theory of operator spaces is very recent. It was developed after Ruan's thesis (1988) by Effros and Ruan and Blecher and Paulsen. It can be described as a noncommutative Banach space theory. An *operator space* is simply a Banach space given together with an isometric linear embedding into the space $B(H)$ of all bounded operators on a Hilbert space H. In this new category, the objects remain Banach spaces but the morphisms become the completely bounded maps (instead of the bounded linear ones). The latter appeared in the early 1980s following Stinespring's pioneering work (1955) and Arveson's fundamental results (1969) on completely positive maps. We study completely bounded (in short c.b.) maps in Chapter 1. This notion became important in the early 1980s through the independent work of Wittstock [Wit1-2], Haagerup [H4], and Paulsen [Pa2]. These authors independently discovered, within a short time interval, the fundamental factorization and extension property of c.b. maps (see Theorem 1.6).

For the reader who might wonder why c.b. maps are the "right" morphisms for the category of operator spaces, here are two arguments that come to mind: Consider $E_1 \subset B(H_1)$ and $E_2 \subset B(H_2)$ and let $\pi\colon B(H_1) \to B(H_2)$ be a C^*-morphism (i.e. a $*$-homomorphism) such that $\pi(E_1) \subset E_2$. Then, quite convincingly, $u = \pi_{|E_1}\colon E_1 \to E_2$ should be an "admissible" morphism in the category of operator spaces. Let us call these morphisms of the "first kind." On the other hand, if a linear map $u\colon E_1 \to E_2$ is of the form $u(x) = VxW$ with $V \in B(H_1, H_2)$ and $W \in B(H_2, H_1)$, then again such an innocent-looking map should be "admissible" and we consider it to be of the "second kind."

But precisely, the factorization theorem of c.b. maps says that any c.b. map $u\colon E_1 \to E_2$ between operator spaces can be written as a composition $E_1 \xrightarrow{u_1} E_3 \xrightarrow{u_2} E_2$ with u_1 of the first kind and u_2 of the second. This is one argument in support of c.b. maps.

Another justification goes via the minimal tensor product: If $E_1 \subset B(H_1)$ and $E_2 \subset B(H_2)$ are operator spaces, their minimal tensor product $E_1 \otimes_{\min} E_2$ is defined as the completion of their algebraic tensor product (denoted by $E_1 \otimes E_2$) with respect to the norm induced on $E_1 \otimes E_2$ by the space $B(H_1 \otimes_2 H_2)$ of all bounded operators on the Hilbertian tensor product $H_1 \otimes_2 H_2$ (this norm coincides with the minimal C^*-norm when E_1 and E_2 are C^*-algebras). Moreover, the isometric embedding

$$E_1 \otimes_{\min} E_2 \subset B(H_1 \otimes_2 H_2)$$

turns $E_1 \otimes_{\min} E_2$ into an operator space. The minimal tensor product is discussed in more detail in §2.1. It is but a natural extension of the "spatial" tensor product of C^*-algebras. In some sense, the minimal tensor product is the most natural operation that is defined using the "operator space structures" of E_1 and E_2 (and not only their norms). This brings us to the second argument supporting the assertion that c.b. maps are the "right" morphisms. Indeed, one can show that a linear map $u\colon E_1 \to E_2$ is c.b. if and only if (iff) for any operator space F the mapping $I_F \otimes u\colon F \otimes_{\min} E_1 \to F \otimes_{\min} E_2$ is bounded in the usual sense. Moreover, the c.b. norm of u could be equivalently defined as

$$\|u\|_{cb} = \sup \|I_F \otimes u\|,$$

where the supremum runs over all possible operator spaces F. (Similarly, u is a complete isometry iff $I_F \otimes u$ is an isometry for all F).

As an immediate consequence, if $v\colon F_1 \to F_2$ is another c.b. map between operator spaces, then $v \otimes u\colon F_1 \otimes_{\min} E_1 \to F_2 \otimes_{\min} E_2$ also is c.b. and

$$\|v \otimes u\|_{CB(F_1 \otimes_{\min} E_1, F_2 \otimes_{\min} E_2)} = \|v\|_{cb}\|u\|_{cb}. \tag{0.1}$$

In conclusion, the c.b. maps are precisely the largest possible class of morphisms for which the minimal tensor product satisfies the "tensorial" property (0.1). So, if one agrees that the minimal tensor product is natural, then one should recognize c.b. maps as the right morphisms.

While the notion of c.b. map (which dates back to the early 1980s, if not sooner) is fundamental to this theory, this new field really took off around 1987 with the thesis of Z. J. Ruan [Ru1], who gave an "abstract characterization" of operator spaces (described in §2.2). Roughly, his result provides a "quantized" counterpart to the norm of a Banach space. When E is an operator space, the norm has to be replaced by the sequence of norms $(\| \ \|_n)$ on the spaces $M_n(E) \cong M_n \otimes_{\min} E$ of all $n \times n$ matrices with entries in E. (The usual norm corresponds to the case $n = 1$.)

In this text, we prefer to replace this sequence of norms by a single one, namely, the norm on the space $\mathcal{K} \otimes_{\min} E$ with $\mathcal{K} = K(\ell_2)$ (= compact operators on ℓ_2). Since $\mathcal{K} = \overline{\cup M_n}$, it is of course very easy to pass from one viewpoint to the other.

The main advantage of Ruan's Theorem is that it allows one to manipulate operator spaces independently of the choice of a "concrete" embedding into $B(H)$. In particular, Ruan's Theorem leads to natural definitions for the dual E^* of an operator space E (independently introduced in [ER2, BP1]) and for the quotient E_1/E_2 of an operator space E_1 by a subspace $E_2 \subset E_1$ (introduced in [Ru1]). These notions are explained in §§2.3 and 2.4. It should be emphasized that they respect the underlying Banach spaces: The dual operator space E^* is the dual Banach space equipped with an additional (specific) operator space structure (i.e., for some $\mathcal{H}$ we have an isometric embedding $E^* \subset B(\mathcal{H})$) and similarly for the quotient space. In addition, the general rules of the duality of Banach spaces (for example, the duality between subspaces and quotients) are preserved in this "new" duality.

More operations can be defined following the same basic idea: complex interpolation (see §2.7) and ultraproducts (see §2.8). We will also use some more elementary constructions, such as direct sums (§2.6), complex conjugates (§2.9), and opposites (§2.10).

Although we described Ruan's thesis as the starting point of the theory, there are many important "prenatal" contributions that shaped this new area. Among them, Christensen and Sinclair's factorization of multilinear maps [CS1] stands out (with its extension to operator spaces by Paulsen and Smith [PaS]). Going back further, there is an important paper by Effros and Haagerup [EH], who discovered that operator spaces may fail the local reflexivity principle, a very interesting phenomenon that is in striking contrast with the Banach space case (their results were inspired by Archbold and Batty's results [AB] for C^*-algebras). Closely connected to theirs, Kirchberg's work on "exact" C^*-algebras (see [Ki1, Wa2]) has also been very influential.

Given a normed space E, there are of course many different ways to embed it into $B(H)$. Two embeddings $j_1\colon E \to B(H_1)$ and $j_2\colon E \to B(H_2)$ are considered equivalent if the associated norms on $M_n(E)$ (or on $\mathcal{K} \otimes E$) are the same for all $n \geq 1$. By an *operator space structure* on E (compatible with the norm) what is meant usually is the data of an equivalence class of such isometric embeddings $j\colon E \to B(H)$ for this equivalence relation.

Blecher and Paulsen [BP1] observed that the set of all operator space structures admissible on a given normed space E has a minimal and a maximal element, which they denoted by $\min(E)$ and $\max(E)$. We summarize their results and various related open questions in Chapter 3.

The minimal tensor product naturally appears as the analog for operator spaces of the "injective" tensor product of Banach spaces. Thus, both Effros-Ruan [ER6–8] and Blecher and Paulsen [BP1] were led to study the operator space analog of the "projective" tensor product in Grothendieck's sense. Recall that a mapping $u\colon E \to F$ (between Banach spaces) is nuclear iff it admits a factorization of the form

$$E \xrightarrow{\alpha} c_0 \xrightarrow{\Delta} \ell_1 \xrightarrow{\beta} F,$$

where α, β, Δ are bounded mappings and Δ is diagonal with coefficients (Δ_n) in ℓ_1.
Moreover, the nuclear norm $N(u)$ is defined as $N(u) = \inf\{\|\alpha\| \sum |\Delta_n|\, \|\beta\|\}$, where the infimum runs over all possible factorizations. As is well known, the space $\mathcal{K} = K(\ell_2)$ of all compact operators on ℓ_2 is the noncommutative analog of c_0, while the space S_1 of all trace class operators on ℓ_2 (with the norm $\|x\|_{S_1} = \mathrm{tr}(|x|)$) is the noncommutative analog of ℓ_1. Now, if E, F are operator spaces, a mapping $u\colon E \to F$ is called "nuclear in the o.s. sense" (introduced in [ER6]) if it admits a factorization of the form

$$E \xrightarrow{\alpha} \mathcal{K} \xrightarrow{\Delta} S_1 \xrightarrow{\beta} F,$$

where α, β are c.b. maps and $\Delta\colon \mathcal{K} \to S_1$ is of the form

$$\Delta(x) = axb$$

with a, b Hilbert-Schmidt. Then the *os*-nuclear norm is defined as

$$N_{os}(u) = \inf\{\|\alpha\|_{cb}\|a\|_2\|b\|_2\|\beta\|_{cb}\}.$$

(Here, of course, $\|a\|_2, \|b\|_2$ denote the Hilbert-Schmidt norms.)

We describe some of the developments of these notions in Chapter 4.

In Banach space theory, Grothendieck's approximation property has played an important role. Recall that Enflo [En] gave the first counterexample in 1972 and Szankowski [Sz] proved around 1980 that the space $B(\ell_2)$ of all bounded operators on ℓ_2 fails the approximation property. Quite naturally, this notion has an operator space counterpart.

In the Banach space case, Grothendieck proved that a space E has the approximation property iff the natural morphism

$$E^* \overset{\wedge}{\otimes} E \to E^* \overset{\vee}{\otimes} E$$

from the projective to the injective tensor product is one to one. We describe in Chapter 4 Effros and Ruan's operator space version of this result.

In Chapter 5, we introduce the Haagerup tensor product $E_1 \otimes_h E_2$ of two operator spaces $E_1,\ E_2$. This notion is of paramount importance in this young theory, and we present it from a somewhat new viewpoint. We prove that if $E_1,\ E_2$ are subspaces of two unital C^*-algebras $A_1,\ A_2$, respectively, then $E_1 \otimes_h E_2$ is naturally embedded (completely isometrically) into the (C^*-algebraic) free product $A_1 * A_2$ ([CES]). We also prove the factorization of completely bounded multilinear maps due to Christensen and Sinclair [CS1] (and to Paulsen and Smith [PaS] for operator spaces) and many more important properties like the self-duality, the shuffle theorem (inspired by [ER10, EKR]), or the embedding of $E_1 \otimes_h E_2$ into the space of maps factoring through the row or column Hilbert space ([ER4]). We also include a brief study of the symmetrized Haagerup tensor product recently introduced in [OP], and we describe the "commutation" between complex interpolation and the Haagerup tensor product ([Ko,P1]).

As an application, we prove in Chapter 6 a characterization of operator algebras due to Blecher-Ruan-Sinclair ([BRS]). The question they answer can be explained as follows: Consider a unital Banach algebra A with a normalized unit and admitting also an operator space structure (i.e., we have $A \subset B(H)$ as a closed linear subspace). When can A be embedded into $B(H)$ as a closed unital subalgebra without changing the operator space structure? They prove that a necessary and sufficient condition is that the product mapping defines a completely contractive map from $A \otimes_h A$ into A. The isomorphic (as opposed to isometric) version of this result was later given by Blecher ([B4]). We include new proofs for these results based on the fact (due to Cole-Lumer-Bernard) that the class of operator algebras (i.e. closed subalgebras of $B(H)$) is stable under quotients by closed ideals. We also give an analogous characterization of operator modules, following [CES].

Curiously, the simplest of all Banach spaces, namely, the Hilbert space ℓ_2, can be realized in many different ways as an operator space. Theoretical physics provides numerous examples of the sort, several of which are described in Chapter 9. Nevertheless, there exists a particular operator space, which we denote by OH, that plays exactly the same central role for operator spaces as the space ℓ_2 among Banach spaces. This space OH is characterized by the property of being canonically completely isometric to its antidual; it also satisfies some remarkable properties with regard to complex interpolation. The space OH is the subject of Chapter 7 (mainly based on [P1]). Since this space gives a nice operator space analog of ℓ_2 or L_2, it is natural to investigate the case of ℓ_p or L_p for $p \neq 2$, as well as the case of a noncommutative vector valued L_p. We do this at the end of Chapter 7, and we return to this in several sections in Chapter 9. However, on that particular topic, we should warn the reader of a certain paradoxical bias: If we do not give to this subject the space it deserves, the sole reason is that we have written an extensive monograph [P2] entirely devoted to it, and we find it easier to refer the reader to the latter for further information.

In Chapter 8 we introduce the group C^*-algebras (full and reduced) and the universal C^*-algebra $C^*\langle E\rangle$ of an operator space E, as well as its universal operator algebra (resp. unital operator algebra) $OA(E)$ (resp. $OA_u(E)$).

Every theory, even one as young as this, displays a collection of "classical" examples that are constantly in the back of the mind of researchers in the field. Our aim in Chapter 9 is to present a preliminary list of such examples for operator spaces. Most of the classical examples of C^*-algebras possess a natural generating subset. Almost always the *linear* span of this subset gives rise to an interesting example of operator space. The discovery that this generating operator space (possibly finite-dimensional) carries a lot of information on the C^*-algebra that it generates has been one of the arguments supporting operator space theory.

In Chapter 9 we make a special effort (directed toward the uninitiated reader) to illustrate the theory with numerous concrete "classical" examples of this type, appearing in various areas of analysis, such as Hankel operators, Fock spaces, and Clifford matrices. Moreover, we describe the linear span of the free unitary generators in the "full" C^*-algebra of the free group (§9.6) as well as in the "reduced" one (§9.7). We emphasize throughout §9 the class of homogeneous Hilbertian operator spaces, and we describe the span of independent Gaussian random variables (or the Rademacher functions) in L_p, in the operator space framework (§9.8). Our treatment underlines the similarity between the latter space and its analog in Voiculescu's free probability theory (see §9.9).

Indeed, it is rather curious that for each $1 \leq p < \infty$ the linear span in L_p of a sequence of independent standard Gaussian variables is completely isomorphic to the span in noncommutative L_p of a free semi-circular (or circular) sequence in Voiculescu's sense (see Theorem 9.9.7). Thus, if we work in L_p with $1 \leq p < \infty$, the operator space structure seems to be roughly the same in the "independent" case and in the "free" one, which is rather surprising.

Our description in §9.8 of the operator space spanned in L_p by Gaussian variables (or the Rademacher functions) is merely a reinterpretation of the noncommutative Khintchine inequalities due to F. Lust-Piquard and the author (see [LuP, LPP]). These inequalities also apply to "free unitaries" (see Theorem 9.8.7) or to "free circular" variables (see Theorem 9.9.7). In view of the usefulness and importance of the classical Khintchine inequalities in commutative harmonic analysis, it is natural to believe that their noncommutative (i.e., operator space theoretic) analog will play an important role in noncommutative L_p-space theory. This is why we have devoted a significant amount of space to this topic in §§9.8 and 9.9. Moreover, in §9.10, we relate these topics to random matrices by showing that the von Neumann algebra of the free group embeds into a (von Neumann sense) ultraproduct of matrix algebras. One can do this by using either the residual finiteness of the free group (as in [Wa1]) or by using one of Voiculescu's matrix models involving independent Gaussian random matrices suitably normalized, and Paul Lévy's concentration of measure phenomenon (see [MS]). Finally, in §9.11, we discuss the possible analogs of Dvoretzky's Theorem for operator spaces (following [P9]).

In Chapter 10 we compare the various examples reviewed in Chapter 9, and we show (by rather elementary arguments) that, except for the few isomorphisms encountered in Chapter 9, these operator spaces are all distinct.

This new theory can already claim some applications to C^*-algebras, many of which are described in the second part of this book. For instance, the existence of an "exotic" C^*-algebra norm on $B(H) \otimes B(H)$ was established in [JP] (see Chapter 22). Moreover, this new ideology allows us to "transfer" into the field of operator algebras several techniques from the "local" (i.e. finite-dimensional) theory of Banach spaces (see Chapter 21). The main applications so far have been to C^*-algebra tensor products. Chapters 11 to 22 are devoted to this topic. We review in Chapters 11 and 12 the basic facts on C^*-norms and nuclear C^*-algebras. Since we are interested in *linear* spaces (rather than cones) of mappings, we strongly emphasize the "decomposable maps" between two C^*-algebras (i.e., those that can be decomposed as a linear combination of (necesssarily at most four) completely positive maps) rather than the completely positive (in short c.p.) ones themselves. Our treatment owes much to Haagerup's landmark paper [H1]. For a more traditional one emphasizing c.p. maps, see [Pa1].

Recall that, if A, B are C^*-algebras, there is a smallest and a largest C^*-norm on $A \otimes B$ and the resulting tensor products are denoted by $A \otimes_{\min} B$ and $A \otimes_{\max} B$. (This notation is coherent with the previous one for the minimal tensor product of operator spaces.) Moreover, a C^*-algebra A is called nuclear if $A \otimes_{\min} B = A \otimes_{\max} B$ for any C^*-algebra B.

In analogy with the Banach space case explored by Grothendieck, the maximal tensor product is projective but not injective, and the minimal one is injective but not projective. Therefore we are naturally led to distinguish two classes: first, the class of C^*-algebras A for which

$$\text{the "functor" } B \to A \otimes_{\max} B \text{ is injective}$$

(this means that $B \subset C$ implies $A \otimes_{\max} B \subset A \otimes_{\max} C$), and, second, the class of C^*-algebras A for which

$$\text{the "functor" } B \to A \otimes_{\min} B \text{ is projective}$$

(this means that $B = C/I$ implies $A \otimes_{\min} B = (A \otimes_{\min} C)/(A \otimes_{\min} I)$). The first class is that of nuclear C^*-algebras reviewed in Chapter 12 (see Exercise 15.2), and the second one is that of exact C^*-algebras studied in Chapter 17.

In Chapter 12 we first give a somewhat new treatment of the well-known equivalences between nuclearity and several forms of approximation properties. We also apply our approach to multilinear maps into a nuclear C^*-algebra (see Theorem 12.11) in analogy with Sinclair's and Smith's recent work [SS3] on injective von Neumann algebras.

Then, Chapter 13 is devoted to a proof of Kirchberg's Theorem, which says that there is a unique C^*-norm on the tensor product of $B(H)$ with the full C^*-algebra of a free group.

The next chapter, Chapter 14, is devoted to an unpublished result of Kirchberg showing that the decomposable maps (i.e. linear combinations of completely positive maps) are the natural morphisms to use if one replaces the minimal tensor product of C^*-algebras by the maximal one. The same proof actually gives a necessary and sufficient condition for a map defined only on a subspace of a C^*-algebra, with range another C^*-algebra, to admit a decomposable extension.

The next two chapters are closely linked together. In Chapters 15 and 16 we present respectively the "weak expectation property" (WEP) and the "local lifting property" (LLP). We start with the C^*-algebra case in connection with Kirchberg's results from Chapter 13, and then we go on to the generalizations to operator spaces. In particular, we study Ozawa's OLLP

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from [Oz3]. At the end of Chapter 16 we discuss at length several equivalent reformulations of Kirchberg's fundamental conjecture on the uniqueness of the C^*-norm on $C^*(\mathbb{F}_\infty) \otimes C^*(\mathbb{F}_\infty)$. For instance, it is the same as asking whether LLP implies WEP.

In Chapter 17 we concentrate on the notion of "exactness" for either operator spaces or C^*-algebras.

Assume that A embedded into $B(H)$ as a C^*-subalgebra. Then A is exact iff $A \otimes_{\min} B$ embeds isometrically into $B(H) \otimes_{\max} B$ for any B. Equivalently, this means that the norm induced on $A \otimes B$ by $B(H) \otimes_{\max} B$ coincides with the min-norm.

This is not the traditional definition of exactness, but it is equivalent to it (see Theorem 17.1). The traditional one is in terms of the exactness of the functor $B \to A \otimes_{\min} B$ in the C^*-category (see (17.1)), and for operator spaces there is also a more appealing reformulation in terms of ultraproducts: Exact operator spaces X are those for which the operation $Y \to Y \otimes_{\min} X$ essentially commutes with ultraproducts (see Theorem 17.7).

The concept of "exactness" owes a lot to Kirchberg's fundamental contributions [Ki1–3]. In particular, Kirchberg proved recently the remarkable definitive result that every separable exact C^*-algebra embeds (as a C^*-subalgebra) into a nuclear one. However, in the operator space framework, the situation is not as clear. When X, Y are operator spaces, we will say that X "locally embeds" into Y if there is a constant C such that, for any finite-dimensional subspace $E \subset X$, there is a subspace $\widetilde{E} \subset Y$ and an isomorphism $u\colon E \to \widetilde{E}$ with $\|u\|_{cb}\|u^{-1}\|_{cb} \leq C$.

We denote

$$d_{SY}(X) = \inf\{C\}, \tag{0.2}$$

that is, $d_{SY}(X)$ is the smallest constant C for which this holds.

With this terminology, an operator space X is exact iff it locally embeds into a nuclear C^*-algebra B. Actually, for such a local embedding we can always take simply $B = \mathcal{K}$ (and $\widetilde{E}$ can be a subspace of M_n with n large enough).

It is natural to introduce (see Chapter 17) the "constant of exactness" $\mathrm{ex}(E)$ of an operator space E (and we will prove in Chapter 17 that it coincides with the just defined constant $d_{S\mathcal{K}}(E)$). This is of particular interest in the finite-dimensional case, and, while many of Kirchberg's results extend to the operator space case, many interesting questions arise concerning the asymptotic growth of these constants for specific E when the dimension of E tends to infinity. (See, e.g., Theorems 21.3 and 21.4.)

In Chapter 18 we describe the main known facts concerning "local reflexivity." While every Banach space is "locally reflexive" (cf. [LiR]), it is not so in the operator space category, and this raises all sorts of interesting questions.

For instance, Kirchberg [Ki1] proved that, for C^*-algebras, exactness implies local reflexivity, but the converse remains open. For operator spaces, we will see, following [EOR], that 1-exact implies 1-locally reflexive (see Theorem 18.21), but the converse is now obviously false since there are reflexive nonexact spaces. We will also show that any "noncommutative L_1-space" (i.e., any predual of a von Neumann algebra) is locally reflexive ([EJR]); we follow the simpler approach of [JLM]. We will describe the properties C, C', and C", which are at the origin of the study of local reflexivity for C^*-algebras ([AB, EH]).

We also return to the OLLP from Chapter 16. We show that the latter for X^{**} implies local reflexivity for X. Moreover, we discuss several interesting conjectures from [Oz3, Oz6] that seem closely related to the old question of whether an ideal in a separable C^*-algebra is automatically the range of a *bounded* linear projection. For instance, it is a very interesting open question of whether the space $B(\ell_2)$ equipped with its maximal operator space structure (in the sense of Chapter 3) is locally reflexive (see also [O3] for related problems).

In another direction, the operator space version of the approximation property (called the OAP, see Definitions 17.11) seems, for C^*-algebras at least, closely related to exactness via the so-called slice map properties (see Corollary 17.14 and the remark below Remark 17.17).

In Chapter 19, we present a version of Grothendieck's factorization theorem adapted to operator spaces, following [JP]. See [PiS] for a different version obtained very recently. In the Banach space context, Grothendieck's theorem implies (see [P4]) that every bounded map $u\colon L^\infty \to L^1$ factors through L^2. Moreover (see [P4, Chapter 9]) the same is true for any bounded map $u\colon A \to B^*$ when A and B are arbitrary C^*-algebras. We prove (see Corollary 19.2) that if $E \subset A$ and $F \subset B$ are exact operator spaces, then any c.b. map $u\colon E \to F^*$ factors through a Hilbert space and the corresponding bilinear form on $E \times F$ extends to a bounded bilinear form on $A \times B$.

In Chapter 21, we prove that for $n > 2$ the metric space OS_n of all n-dimensional operator spaces is not separable for its natural metric, in sharp contrast to the Banach space analog. We give two approaches to this key result, one based on the factorization from Chapter 19 and one based on a specific constant $C(n)$ studied in Chapter 20. This constant quantifies a certain asymptotic phenomenon for n-tuples of unitary $N \times N$ matrices when the size N tends to infinity. The proof that $C(n) < n$ involves surprisingly deep ingredients (Property T, expanders, random matrices), which are described in Chapter 20.

In answer to a question of Kirchberg, it was proved in [JP] that $B(H) \otimes_{\min} B(H) \neq B(H) \otimes_{\max} B(H)$. This is described in Chapter 22. An important role is played behind the scene by the full C^*-algebra of the free group $\mathbb{F}_\infty$ on infinitely many generators. This C^*-algebra is denoted by $C^*(\mathbb{F}_\infty)$. The ideas involved emphasize the importance of a subclass among operator spaces, namely, those spaces E such that any of their finite-dimensional subspaces embed almost completely isometrically into $C^*(F_\infty)$. These are the spaces satisfying $d_{SC^*(F_\infty)}(E) = 1$ in notation (0.2). The constant $d_{SC^*(F_\infty)}(E)$ is abbreviated to $d_f(E)$ in Chapter 22. In the finite-dimensional case, these spaces form a separable subclass, for a natural metric, of the class (itself non-separable) of all finite-dimensional operator spaces. Moreover, this subclass is stable under duality and various tensor products. It turns out that many of the questions that have been examined for the C^*-algebra $\mathcal{K}$ (of all compact operators on ℓ_2) in connection with exactness have interesting analogs for the C^*-algebra $C^*(F_\infty)$ (see Chapter 22).

Given the interplay between C^*-algebras and operator spaces, it is natural to ask: If two C^*-algebras A_1 and A_2 are isomorphic as operator spaces (i.e., completely isomorphic), are they isomorphic as C^*-algebras? The answer is negative. However, it turns out that A_1 and A_2 must share numerous C^*-properties such as nuclearity, exactness, WEP, and injectivity (for von Neumann algebras). These questions are discussed in Chapter 23.

In Chapter 24 (mainly a survey) we study injective and projective operator spaces.

In the third part of the book, we concentrate on *non-self-adjoint* operator algebras. The typical examples are algebras of bounded analytic functions on some domain, the subalgebra of $B(\ell_2)$ formed of all triangular matrices (and more generally the so-called nest algebras [Da1]), or the unital algebra generated by a single operator T in $B(\ell_2)$. Their behavior is usually quite different from that of C^*-algebras.

In Chapter 25 we return to the study of the maximal tensor product, already considered mainly for C^*-algebras in Chapters 11 and 12. Here we study more generally the maximal tensor product $A_1 \otimes_{\max} A_2$, when A_1 and A_2 are two unital operator algebras. This was first investigated in [PaP]. We discuss the analogue of nuclearity for unital operator algebras and various related questions. Our results are closely related to Haagerup's results [H1] on the relation between the decomposability properties of c.b. maps into a C^*-algebra A and the nuclearity of A, but our approach seems new. A bit surprisingly, it turns out that several basic facts remain valid for non-self-adjoint operator algebras. For instance, a unital operator algebra A satisfies $B \otimes_{\min} A = B \otimes_{\max} A$ (isomorphically) for any unital operator algebra B iff the identity on A is approximable by finite-rank decomposable maps in a suitable way (see Theorem 25.9). Since we allow isomorphism (and not only isometry) in $B \otimes_{\min} A = B \otimes_{\max} A$, there are clearly non-self-adjoint examples satisfying this (for instance, finite-dimensional quotients of the disc algebra, as discussed in Example 25.6); however, in the isometric case, we prove that only self-adjoint algebras can satisfy this (see Theorem 25.11). Further results are given in [LeM4].