Harmonic maps, conservation laws and moving frames

Second edition

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CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK
40 West 20th Street, New York, NY 10011-4211, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
Ruiz de Alarcón 13, 28014, Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa

http://www.cambridge.org

The first edition of this work was published in French as *Applications harmoniques, lois de conservation et repères mobiles* by Diderot Editeur in 1996. An English translation, translated by L. Almeida, was published by Diderot Editeur in 1997.

Second edition published 2002

Printed in the United Kingdom at the University Press, Cambridge

A catalogue record of this book is available from the British Library

ISBN 0 521 81160 0 hardback

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Geometric and analytic setting

This chapter essentially describes the objects and properties that will interest us in this work. For a more detailed exposition of the general background in Riemannian geometry and in analysis on manifolds, one may refer for instance to [183] and [98]. After recalling how to associate, to each Riemannian metric on a manifold, a Laplacian operator on the same manifold, we will give a definition of smooth harmonic map between two manifolds. Very soon, we will use the variational framework, which consists in viewing harmonic maps as the critical points of the Dirichlet functional.

Next, we introduce a frequently used ingredient in this book: Noether's theorem. We present two versions of it: one related to the symmetries of the image manifold, and the other which is a consequence of an invariance of the problem under diffeomorphisms of the domain manifold (in this case it is not exactly Noether's theorem, but a "covariant" version).

These concepts may be extended to contexts where the map between the two manifolds is less regular. In fact, a relatively convenient space is that of maps with finite energy (Dirichlet integral), $H^1(\mathcal{M},\mathcal{N})$. This space appears naturally when we try to use variational methods to construct harmonic maps, for instance the minimization of the Dirichlet integral. The price to pay is that when the domain manifold has dimension larger than or equal to 2, maps in $H^1(\mathcal{M},\mathcal{N})$ are not smooth, in general. Moreover, $H^1(\mathcal{M},\mathcal{N})$ does not have a differentiable manifold structure. This yields that several non-equivalent generalizations of the notion of harmonic function coexist in $H^1(\mathcal{M},\mathcal{N})$ (weakly harmonic, stationary harmonic, minimizing, ...). We will conclude this chapter with a brief survey of the known results on weakly harmonic maps in $H^1(\mathcal{M},\mathcal{N})$. As we will see, the results are considerably different according to which definition of critical point of the Dirichlet integral we adopt.

NOTATION: $\mathcal{M}$ and $\mathcal{N}$ are differentiable manifolds. Most of the time, $\mathcal{M}$ plays the role of domain manifold, and $\mathcal{N}$ that of image manifold; we will suppose $\mathcal{N}$ to be compact without boundary. In case they are abstract manifolds (and not submanifolds) we may suppose that they are $\mathcal{C}^\infty$ (in fact, thanks to a theorem of Whitney, we may show that every $\mathcal{C}^1$ manifold is $\mathcal{C}^1$-diffeomorphic to a $\mathcal{C}^\infty$ manifold). Unless stated otherwise, $\mathcal{M}$ is equipped with a $\mathcal{C}^{0,\alpha}$ Riemannian metric g, where $0 < \alpha < 1$. For $\mathcal{N}$, we consider two possible cases: either it is an abstract manifold with a $\mathcal{C}^1$ Riemannian metric h, or we will need to suppose it is a $\mathcal{C}^2$ immersed submanifold of $\mathbb{R}^N$. The second situation is a special case of the first one, but nevertheless, Nash's theorem (see [123], [74] and [77]) assures us that if h is $\mathcal{C}^l$ for $l \geq 3$, then there exists a $\mathcal{C}^l$ isometric immersion of $(\mathcal{N}, h)$ in $(\mathbb{R}^N, \langle ., . \rangle)$.

Several regularity results are presented in this book. We will try to present them under minimal regularity hypotheses on $(\mathcal{M}, g)$ and $(\mathcal{N}, h)$, keeping in mind that any improvement of the hypotheses on $(\mathcal{M}, g)$ and $(\mathcal{N}, h)$ automatically implies an improvement of the conclusion, as explained in theorem 1.5.1.

We write $m := \dim \mathcal{M}$ and $n := \dim \mathcal{N}$.

1.1 The Laplacian on $(\mathcal{M}, g)$

For every metric g on $\mathcal{M}$ there exists an associated Laplacian operator Δ_g, acting on all smooth functions on $\mathcal{M}$ taking their values in $\mathbb{R}$ (or any vector space over $\mathbb{R}$ or $\mathbb{C}$). To define it, let us use a local coordinate system $(x^1, \ldots, x^m)$ on $\mathcal{M}$. Denote by

$$g_{\alpha\beta}(x) = g(x)\left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}\right)$$

the coefficients of the metric, and by $\det g(x)$ the determinant of the matrix whose elements are $g_{\alpha\beta}(x)$. Then, for each real-valued function ϕ defined over an open subset Ω of $\mathcal{M}$, we let

$$\Delta_g \phi = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^\alpha}\left(\sqrt{\det g}\, g^{\alpha\beta}(x) \frac{\partial \phi}{\partial x^\beta}\right) \tag{1.1}$$

where we adopt the convention that repeated indices should be summed over. The metric g induces on the cotangent space $T_x^* \mathcal{M}$ a metric which we denote by $g^\sharp$. Its coefficients are given by $g^{\alpha\beta} = g^\sharp(dx^\alpha, dx^\beta)$. Recall that $g^{\alpha\beta}(x)$ represents an element of the inverse matrix of $(g_{\alpha\beta})$.

Definition 1.1.1 *Any smooth function ϕ defined over an open subset Ω of $\mathcal{M}$ and satisfying*

$$\Delta_g \phi = 0$$

is called a harmonic function.

We can easily check through a computation that the operator Δ_g does not depend on the choice of the coordinate system, but it will be more pleasant to obtain this as a consequence of a variational definition of Δ_g. Let

$$d\mathrm{vol}_g = \sqrt{\det g(x)}\, dx^1 \dots dx^m, \tag{1.2}$$

be the Riemannian measure. For each smooth function ϕ from $\Omega \subset \mathcal{M}$ to $\mathbb{R}$, let

$$E_{(\Omega,g)}(\phi) = \int_\Omega e(\phi)\ d\mathrm{vol}_g \tag{1.3}$$

be the energy or Dirichlet integral of ϕ (which may be finite or not). Here, $e(\phi)$ is the energy density of ϕ and is given by

$$e(\phi) = \frac{1}{2} g^{\alpha\beta}(x) \frac{\partial \phi}{\partial x^\alpha} \frac{\partial \phi}{\partial x^\beta}. \tag{1.4}$$

It is easy to check that the Dirichlet integral does not depend on the choice of the local coordinate system and that, if ψ is a compactly supported smooth function on $\Omega \subset \mathcal{M}$, then for all $t \in \mathbb{R}$,

$$E_{(\Omega,g)}(\phi + t\psi) = E_{(\Omega,g)}(\phi) - t \int_\Omega (\Delta_g \phi)\psi\ d\mathrm{vol}_g + O(t^2). \tag{1.5}$$

Hence, $-\Delta_g$ appears as the variational derivative of E_Ω, which provides us with an equivalent definition of the Laplacian.

Thus, the Laplacian does not depend on the coordinate system used. However, it depends on the metric. For instance, let us consider the effect of a conformal transformation on $(\mathcal{M}, g)$, i.e. compare the Dirichlet integrals and the Laplacians on the manifolds $(\mathcal{M}, g)$ and $(\mathcal{M}, e^{2v} g)$, where v is a smooth real-valued function on $\mathcal{M}$. We have

$$d\mathrm{vol}_{e^{2v} g} = e^{mv}\ d\mathrm{vol}_g, \tag{1.6}$$

and for the energy density (1.4)

$$e_{e^{2v}g}(\phi) = e^{-2v} e_g(\phi). \tag{1.7}$$

Thus,

$$E_{(\Omega, e^{2v}g)}(\phi) = \int_\Omega e^{(m-2)v} e_g(\phi)\, d\mathrm{vol}_g. \tag{1.8}$$

However, we notice that in case $m = 2$, the Dirichlet integrals calculated using the metrics g and $e^{2v}g$ coincide, and thus are invariant under a conformal transformation of the metric.

Still in the case $m = 2$, we have

$$\Delta_{e^{2v}g}(\phi) = e^{-2v} \Delta_g \phi\,. \tag{1.9}$$

Therefore, for $m = 2$, every function which is harmonic over $(\mathcal{M}, g)$ will also be so over $(\mathcal{M}, e^{2v}g)$. More generally, if $(\mathcal{M}, g)$ and $(\mathcal{M}', g')$ are two Riemannian surfaces and Ω and Ω' are two open subsets of $\mathcal{M}$ and $\mathcal{M}'$ respectively, then if $T : (\Omega, g) \longrightarrow (\Omega', g')$ is a conformal diffeomorphism, we have

$$E_{(\Omega,g)}(\phi \circ T) = E_{(\Omega',g')}(\phi), \forall \phi \in \mathcal{C}^1(\Omega', \mathbb{R}) \tag{1.10}$$

and

$$\Delta_g(\phi \circ T) = \lambda(\Delta_{g'}\phi) \circ T, \tag{1.11}$$

where

$$\lambda = \frac{1}{2} g^{\alpha\beta}(x) g'_{ij}(T(x)) \frac{\partial T^i}{\partial x^\alpha} \frac{\partial T^j}{\partial x^\beta}\,.$$

Thus,

Proposition 1.1.2 *The Dirichlet integral, and the set of harmonic functions over an open subset of a Riemannian surface, depend only on the conformal structure of this surface.*

This phenomenon, characteristic of dimension 2, has many consequences, among them the following, which is very useful: first recall that according to the theorem below, locally all conformal structures are equivalent.

Theorem 1.1.3 *Let $(\mathcal{M}, g)$ be a Riemannian surface. Then, for each point x_0 in $(\mathcal{M}, g)$, there is a neighborhood U of x_0 in $\mathcal{M}$, and a diffeomorphism T from the disk*

$$D = \{(x, y) \in \mathbb{R}^2 |\ x^2 + y^2 < 1\}$$

to U, such that, if c is the canonical Euclidean metric on the disk, $T : (D, c) \longrightarrow (U, g)$ is a conformal map. We say that T^{-1} is a local conformal chart in $(\mathcal{M}, g)$ and that (x, y) are conformal coordinates.

Remark 1.1.4 *There are several proofs of this result, depending on the regularity of g. The oldest supposes g to be analytic. Later methods like that of S.S. Chern (see [36]), where g is supposed to be just Hölder continuous, have given results that are valid under weaker regularity assumptions. At the end of this book (theorem 5.4.3) we can find a proof of theorem 1.1.3 under weaker assumptions.*

Using theorem 1.1.3, we can express the Dirichlet integral over U of a map ϕ from $\mathcal{M}$ to $\mathbb{R}$, simply as

$$\int_U e(\phi)\ d\mathrm{vol}_g = \int_{D^2} \frac{1}{2} \left[\left(\frac{\partial(\phi \circ T)}{\partial x} \right)^2 + \left(\frac{\partial(\phi \circ T)}{\partial y} \right)^2 \right] dxdy\,,$$

and ϕ will be harmonic if and only if

$$\Delta(\phi \circ T) = \frac{\partial^2(\phi \circ T)}{\partial x^2} + \frac{\partial^2(\phi \circ T)}{\partial y^2} = 0\,.$$

Thus, when studying harmonic functions on a Riemannian surface, we can always suppose, at least locally, that our equations are similar to those corresponding to the case where the domain metric is flat (Euclidean).

1.2 Harmonic maps between two Riemannian manifolds

We now introduce a second Riemannian manifold, $\mathcal{N}$, supposed to be compact and without boundary, which we equip with a metric h. Recall that over any Riemannian manifold $(\mathcal{N}, h)$, there exists a unique *connection* or *covariant derivative*, ∇, having the following properties.

(i) ∇ is a linear operator acting on the set of smooth (at least $\mathcal{C}^1$) tangent vector fields on $\mathcal{N}$. To each $\mathcal{C}^k$ vector field X (where

$k \geq 1$) on $\mathcal{N}$, we associate a field of $\mathcal{C}^{k-1}$ linear maps from $T_y\mathcal{N}$ to $T_y\mathcal{N}$ defined by

$$T_y\mathcal{N} \ni Y \longmapsto \nabla_Y X \in T_y\mathcal{N}.$$

(ii) ∇ is a derivation, i.e. for any smooth function α from $\mathcal{N}$ to $\mathbb{R}$, any vector field X and any vector Y in $T_y\mathcal{N}$,

$$\nabla_Y(\alpha X) = \mathrm{d}\alpha(Y)X + \alpha\nabla_Y X.$$

(iii) The metric h is parallel for ∇, i.e. for any vector fields X, Y, and for any vector Z in $T_y\mathcal{N}$,

$$\mathrm{d}(h_y(X,Y))(Z) = h_y(\nabla_Z X, Y) + h_y(X, \nabla_Z Y).$$

(iv) ∇ has zero torsion, i.e. for any vector fields X, Y,

$$\nabla_X Y - \nabla_Y X - [X,Y] = 0\,.$$

∇ is called the Levi-Civita connection.

Let $(y^1, \ldots, y^n)$ be a local coordinate system on $\mathcal{N}$, and $h_{ij}(y)$ the coefficients of the metric h in these coordinates. We can show (see, for instance, [47]) that for any vector field $Y = Y^i \frac{\partial}{\partial y^i}$,

$$\nabla_X \left(Y^i \frac{\partial}{\partial y^i} \right) = \left(X^j \frac{\partial\, Y^i}{\partial y^j} + \Gamma^i_{jk} X^j Y^k \right) \frac{\partial}{\partial y^i}$$

where

$$\Gamma^i_{jk} = \frac{1}{2} h^{il} \left(\frac{\partial h_{jl}}{\partial x^k} + \frac{\partial h_{kl}}{\partial x^j} - \frac{\partial h_{jk}}{\partial x^l} \right) \tag{1.12}$$

are the Christoffel symbols.

Let $u : \mathcal{M} \longrightarrow \mathcal{N}$ be a smooth map.

Definition 1.2.1 *u is a harmonic map from $(\mathcal{M}, g)$ to $(\mathcal{N}, h)$ if and only if u satisfies at each point x in $\mathcal{M}$ the equation*

$$\Delta_g u^i + g^{\alpha\beta}(x)\Gamma^i_{jk}(u(x)) \frac{\partial u^j}{\partial x^\alpha} \frac{\partial u^k}{\partial x^\beta} = 0\,. \tag{1.13}$$

Once more, the reader may check that this definition is independent of the coordinates chosen on $\mathcal{M}$ and $\mathcal{N}$. However, it is easier to see this once we notice that harmonic maps are critical points of the Dirichlet functional

$$E_{(\mathcal{M},g)}(u) = \int_{\mathcal{M}} e(u)(x)\, d\mathrm{vol}_g, \tag{1.14}$$

where

$$e(u)(x) = \frac{1}{2} g^{\alpha\beta}(x) h_{ij}(u(x)) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta},$$

and where u is forced to take its values in the manifold $\mathcal{N}$. The proof of this result, in a more general setting, will be given later on, in lemma 1.4.10. When we say that $u : \mathcal{M} \longrightarrow \mathcal{N}$ is a critical point of $E_{(\mathcal{M},g)}$, it is implicit that for each one-parameter family of deformations

$$u_t : \mathcal{M} \longrightarrow \mathcal{N},\ t \in I \subset \mathbb{R},$$

which has a $\mathcal{C}^1$ dependence on t, and is such that $u_0 \equiv u$ on $\mathcal{M}$ and, for every t, $u_t = u$ outside a compact subset K of $\mathcal{M}$, we have

$$\lim_{t \to 0} \frac{E_{(\mathcal{M},g)}(u_t) - E_{(\mathcal{M},g)}(u)}{t} = 0.$$

Different types of deformations will be specified in section 1.4. Notice that, by checking that $E_{(\mathcal{M},g)}(u)$ is invariant under a change of coordinates on $(\mathcal{M}, g)$, we show that definition 1.2.1 does not depend on the coordinates chosen on $\mathcal{M}$ (the same is true for the coordinates on $\mathcal{N}$).

Effect of a conformal transformation on $(\mathcal{M}, g)$, if $m = 2$

As we noticed in the previous section, in dimension 2 (i.e. when $\mathcal{M}$ is a surface), the Dirichlet functional for real-valued functions on $\mathcal{M}$ is invariant under conformal transformations of $(\mathcal{M}, g)$. This property remains true when we replace real-valued functions by maps into a manifold $(\mathcal{N}, h)$. An immediate consequence of this is the following generalization of proposition 1.1.2.

Proposition 1.2.2 *The Dirichlet integral, and the set of harmonic maps on an open subset of a Riemannian surface, depend only on the conformal structure.*

By theorem 1.1.3, we can always suppose that we have locally conformal coordinates $(x, y) \in \mathbb{R}^2$ on $(\mathcal{M}, g)$. In these coordinates equation (1.13) becomes

$$\frac{\partial^2 u^i}{\partial x^2} + \frac{\partial^2 u^i}{\partial y^2} + \Gamma^i_{jk}(u)\left(\frac{\partial u^j}{\partial x}\frac{\partial u^k}{\partial x} + \frac{\partial u^j}{\partial y}\frac{\partial u^k}{\partial y}\right) = 0.$$

Another definition

Henceforth, we will not use formulation (1.13), but an alternative one where we think of $\mathcal{N}$ as a submanifold of a Euclidean space. In fact, thanks to the Nash–Moser theorem ([123], [102], [77]), we know that, provided h is $\mathcal{C}^3$, it is always possible to isometrically embed $(\mathcal{N}, h)$ into a vector space $\mathbb{R}^N$, with the Euclidean scalar product $\langle .\,, .\rangle$. Then, we will obtain a new expression for the Dirichlet integral

$$E_{(\mathcal{M},g)}(u) = \int_{\mathcal{M}} \frac{1}{2} g^{\alpha\beta}(x)\left\langle \frac{\partial u}{\partial x^\alpha}, \frac{\partial u}{\partial x^\beta}\right\rangle d\mathrm{vol}_g \tag{1.15}$$

where now we think of u as a map from $\mathcal{M}$ to $\mathbb{R}^N$ satisfying the constraint

$$u(x) \in \mathcal{N}, \forall x \in \mathcal{M}. \tag{1.16}$$

Therefore, we have another definition.

Definition 1.2.3 *u is a harmonic map from $(\mathcal{M}, g)$ to $\mathcal{N} \subset \mathbb{R}^N$, if and only if u is a critical point of the functional defined by (1.15), among the maps satisfying the constraint (1.16). We can then see that u satisfies*

$$\Delta_g u \perp T_{u(x)}\mathcal{N},\ \forall x \in \mathcal{M}\,. \tag{1.17}$$

The proof of (1.17) will be given, in a more general setting, in lemma 1.4.10. This equation means that for every point x of $\mathcal{M}$, $\Delta_g u(x)$ is a vector of $\mathbb{R}^N$ belonging to the normal subspace to $\mathcal{N}$ at $u(x)$. At first glance, condition (1.17) seems weaker than equation (1.13), since we just require that the vector $\Delta_g u$ belongs to a subspace of $\mathbb{R}^N$. This imprecision is illusory: by this we mean that it is possible to calculate the normal component of $\Delta_g u$, a priori unknown, using the first derivatives of u.

Lemma 1.2.4 *Let u be a $\mathcal{C}^2$ map from $\mathcal{M}$ to $\mathcal{N}$, not necessarily harmonic. For each $x \in \mathcal{M}$, let $P_u^\perp$ be the orthogonal projection from $\mathbb{R}^N$ onto the normal subspace to $T_{u(x)}\mathcal{N}$ in $\mathbb{R}^N$. Then, for every x in $\mathcal{M}$,*

$$P_u^\perp(\Delta_g u) = -g^{\alpha\beta} A(u)\left(\frac{\partial u}{\partial x^\alpha}, \frac{\partial u}{\partial x^\beta}\right), \tag{1.18}$$

where $A(y)$ is an $\mathbb{R}^N$-valued symmetric bilinear form on $T_y\mathcal{N}$ whose coefficients are smooth functions of y. A is the second fundamental form of the embedding of $\mathcal{N}$ into $\mathbb{R}^N$.

A first way of writing A explicitly is to choose over sufficiently small open sets ω of $\mathcal{N}$ an $(N-n)$-tuple of smooth vector fields $(e_{n+1}, \dots, e_N) : \omega \longrightarrow (\mathbb{R}^N)^{N-n}$, such that at each point $y \in \omega, (e_{n+1}(y), \dots, e_n(y))$ is an orthonormal basis of $(T_y\mathcal{N})^\perp$. Then, for each pair of vectors (X, Y) in $(T_y\mathcal{N})^2$,

$$A(y)(X, Y) = \sum_{j=n+1}^{N} \langle X, D_Y e_j\rangle e_j \,,$$

where $D_Y e_j = \sum_{i=1}^N Y^i \frac{\partial e_j}{\partial y^i}$ is the derivative of e_j along Y in $\mathbb{R}^N$. Another possible definition for A is

$$A(y)(X, Y) = D_X P_y^\perp(Y)\,. \tag{1.19}$$

Proof of lemma 1.2.4 We have

$$P_u^\perp\left(g^{\alpha\beta}\sqrt{\det g}\,\frac{\partial u}{\partial x^\beta}\right) = 0\,,$$

which implies that

$$P_u^\perp\left(\frac{\partial}{\partial x^\alpha}\left(g^{\alpha\beta}\sqrt{\det g}\,\frac{\partial u}{\partial x^\beta}\right)\right) + \frac{\partial P_u^\perp}{\partial x^\alpha}\left(g^{\alpha\beta}\sqrt{\det g}\,\frac{\partial u}{\partial x^\beta}\right) = 0\,.$$

Thus,

$$\begin{aligned} P_u^\perp(\Delta_g u) &= \frac{1}{\sqrt{\det g}} P_u^\perp\left(\frac{\partial}{\partial x^\alpha}\left(g^{\alpha\beta}\sqrt{\det g}\,\frac{\partial u}{\partial x^\beta}\right)\right) \\ &= -g^{\alpha\beta} D_{\frac{\partial u}{\partial x^\alpha}} P_u^\perp\left(\frac{\partial u}{\partial x^\beta}\right). \end{aligned}$$

And we conclude that

$$P_u^{\perp}(\Delta_g u) = -g^{\alpha\beta}A(u)\left(\frac{\partial u}{\partial x^\alpha}, \frac{\partial u}{\partial x^\beta}\right), \tag{1.20}$$

where A is given by (1.19). □

We come back to harmonic maps according to definition 1.2.3 and denote, for each $y \in \mathcal{N}$, by P_y the orthogonal projection of $\mathbb{R}^N$ onto $T_y\mathcal{N}$. Since $P_y + P_y^{\perp} = \mathbb{1}$, from lemma 1.2.4 we deduce that for every harmonic map u from $(\mathcal{M}, g)$ to $\mathcal{N}$,

$$\Delta_g u + g^{\alpha\beta}A(u)\left(\frac{\partial u}{\partial x^\alpha}, \frac{\partial u}{\partial x^\beta}\right) = 0. \tag{1.21}$$

Example 1.2.5 $\mathbb{R}^n$-VALUED MAPS
If the image manifold is a Euclidean vector space, such as $(\mathbb{R}^n, \langle ., . \rangle)$, then a map $u : (\mathcal{M}, g) \longrightarrow \mathbb{R}^n$ is harmonic if and only if each of its components u^i is a real-valued harmonic function on $(\mathcal{M}, g)$.

Example 1.2.6 GEODESICS
If the domain manifold $\mathcal{M}$ has dimension 1 (i.e. is either an interval in $\mathbb{R}$, or a circle), equation (1.21) becomes, denoting by t the variable on $\mathcal{M}$,

$$\frac{d^2u}{dt^2} + A(u)\left(\frac{du}{dt}, \frac{du}{dt}\right) = 0\,,$$

which is the equation satisfied by a constant speed parametrization of a geodesic in $(\mathcal{N}, h)$.

Example 1.2.7 MAPS TAKING THEIR VALUES IN THE UNIT SPHERE OF $\mathbb{R}^3$
In this case we have

$$\mathcal{N} = S^2 = \{y \in \mathbb{R}^3 | \, |y| = 1\},$$

where $|y| = \left(\sum_{i=1}^{3} (y^i)^2\right)^{\frac{1}{2}}$ is the norm of y. Notice that for each map $u : (\mathcal{M}, g) \longrightarrow S^2$, we have

$$0 = \Delta_g |u|^2 = 2\,\langle u, \Delta_g u\rangle + 4e(u),$$

where

$$e(u) = \frac{1}{2} g^{\alpha\beta}(x) \left\langle \frac{\partial u}{\partial u^\alpha}, \frac{\partial u}{\partial x^\beta} \right\rangle,$$

and thus, since the normal space to S^2 at u is $\mathbb{R}u$,

$$\begin{aligned} P_u^\perp(\Delta_g u) &= \langle u, \Delta_g u \rangle \, u \\ &= -2e(u)u\,. \end{aligned} \tag{1.22}$$

But if u is an S^2-valued harmonic map,

$$P_u(\Delta_g u) = 0\,,$$

which, together with (1.22), yields

$$\Delta_g u + 2e(u)u = 0. \tag{1.23}$$

Exercises

1.1 Let $(\mathcal{M}, g)$ be a Riemannian manifold. Show that a map $u : (\mathcal{M}, g) \longrightarrow S^1$ is harmonic if and only if for any simply connected subset Ω of $\mathcal{M}$, there exists a harmonic function $f : (\Omega, g) \longrightarrow \mathbb{R}$ such that

$$u(x) = e^{if(x)}, \ \forall x \in \Omega.$$

1.3 Conservation laws for harmonic maps

Noether's theorem is a very general result in the calculus of variations. It enables us to construct a divergence-free vector field on the domain space, from a solution of a variational problem, provided we are in the presence of a continuous symmetry. For 1–dimensional variational problems, the divergence-free vector field is just a quantity which is conserved in time (= the variable): for instance in mechanics, the energy or the momentum. We can find in [124] a presentation of Noether's theorem, and of its extensions (see also [140]). We will present here two versions which are frequently used for harmonic maps. The first is obtained in the case where the symmetry group acts on the image manifold, and the second is connected to the symmetries of the domain.

1.3.1 Symmetries on $\mathcal{N}$

We start with a simple example, that of harmonic maps from an open set Ω of $\mathbb{R}^m$, taking values in the sphere $S^2 \subset \mathbb{R}^3$. In this case equation (1.23) of the previous section can be written as

$$\Delta u + u|du|^2 = 0, \tag{1.24}$$

where

$$|du|^2 = \sum_{i=1}^{3}\sum_{\alpha=1}^{m}\left(\frac{\partial u^i}{\partial x^\alpha}\right)^2 = 2e(u),$$

and

$$\Delta u = \sum_{\alpha=1}^{m}\frac{\partial^2 u}{(\partial x^\alpha)^2}.$$

Taking the vector product of equation (1.24) by u, we obtain

$$u \times \Delta u = 0\,, \tag{1.25}$$

or equivalently,

$$\sum_{\alpha=1}^{m}\frac{\partial}{\partial x^\alpha}\left(u \times \frac{\partial u}{\partial x^\alpha}\right) = \sum_{\alpha=1}^{m}\frac{\partial u}{\partial x^\alpha} \times \frac{\partial u}{\partial x^\alpha} + u \times \Delta u = 0\,. \tag{1.26}$$

Several authors (see [99], [35], [153]) have noticed and used independently equation (1.26). Its interest is that it singles out a vector field $j = (j^1, \ldots, j^m)$ on Ω, given by

$$j^\alpha = u \times \frac{\partial u}{\partial x^\alpha}$$

which is divergence-free. It allows us to rewrite (1.24) in the form (1.26) where the derivatives of u appear in a linear and not quadratic way, which is extremely useful when we are working with weak regularity hypotheses on the solution (see the works of Luís Almeida [2] and Yuxin Ge [65] for an example of how to take advantage of this equation under very weak regularity hypotheses).

We will now see that the existence of this conservation law ($\operatorname{div} j = 0$) is a general phenomenon, which is due to the symmetries of the sphere S^2. Let Ω be an open subset of $\mathcal{M}$ and L be a Lagrangian defined for maps from Ω to $\mathcal{N}$: we suppose that L is a $\mathcal{C}^1$ function defined on

$T\mathcal{N} \otimes_{\mathcal{M}\times\mathcal{N}} T^*\mathcal{M} := \{(x, y, A) \mid (x, y) \in \mathcal{M}\times\mathcal{N},\ A \in T_y\mathcal{N} \otimes T_x^*\mathcal{M}\}$ with values in $\mathbb{R}$ (here A can be seen as a linear map from $T_x\mathcal{M}$ to $T_y\mathcal{N}$). We take a ($\mathcal{C}^1$ density) measure $d\mu(x)$ on Ω. It is then possible to define a functional $\mathcal{L}$ on the set of maps $\mathcal{C}^1(\Omega, \mathcal{N})$ by letting

$$\mathcal{L}(u) = \int_\Omega L(x, u(x), du(x))d\mu(x)\,.$$

For instance, if we are given metrics g and h over $\mathcal{M}$ and $\mathcal{N}$, we may choose

$$L(x, u(x), du(x)) = e(u)(x) = \frac{1}{2} g^{\alpha\beta}(x) h_{ij}(u(x)) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta}$$

and

$$d\mu(x) = \ d\mathrm{vol}_g(x).$$

Let X be a tangent vector field on $\mathcal{N}$. We say that X is an infinitesimal symmetry for L if and only if

$$\frac{\partial L}{\partial y^i}(x, y, A) X^i(y) + \frac{\partial L}{\partial A^i_\alpha}(x, y, A) \frac{\partial X^i}{\partial y^j}(y) A^j_\alpha = 0\,. \tag{1.27}$$

This implies, in particular, the following relation. Suppose that the vector field X is Lipschitz. It is then possible to integrate the flow of X for all time ($\mathcal{N}$ is compact!). For any $y \in \mathcal{N}$, $t \in \mathbb{R}$, write

$$\exp_y tX = \gamma(t) \in \mathcal{N}\,,$$

where γ is the solution of

$$\left\{ \begin{array}{lcl} \gamma(0) & = & y \\ \dfrac{d\gamma}{dt} & = & X(\gamma)\,. \end{array} \right.$$

A consequence of (1.27) is that for every map u from Ω to $\mathcal{N}$,

$$L(x, \exp_u tX, d(\exp_u tX)) = L(x, u, du)\,. \tag{1.28}$$

To check it, it suffices to differentiate equation (1.28) w.r.t. t and use (1.27).

Theorem 1.3.1 *Let X be a Lipschitz tangent vector field on $\mathcal{N}$, which is an infinitesimal symmetry for L. If $u : \Omega \longrightarrow \mathcal{N}$ is a critical point of $\mathcal{L}$, then*

$$\operatorname{div}\left(\frac{\partial L}{\partial A} \cdot X(u)\right) = 0 \tag{1.29}$$

or equivalently, using the coordinates $(x^1, \ldots, x^m)$ on Ω such that $d\mu = \rho(x)dx^1 \ldots dx^m$,

$$\sum_{\alpha=1}^{m} \frac{\partial}{\partial x^\alpha}\left(\rho(x) X^i(u) \frac{\partial L}{\partial A^i_\alpha}(x, u, du)\right) = 0\,. \tag{1.30}$$

Remark 1.3.2

(i) *In the case where L is the Lagrangian of the harmonic map, this result was first obtained in [134].*

(ii) *The vector field J defined over Ω by*

$$J^\alpha = \rho(x) \frac{\partial L}{\partial A^i_\alpha}(x, u, du) X^i(u)$$

is often called the Noether current by physicists. Equations (1.29) and (1.30) are called conservation laws.

Proof of theorem 1.3.1 To lighten the notation, we can always suppose that the coordinates $(x^1, \ldots, x^m)$ on Ω are such that $d\mu = dx^1 \ldots dx^m$. This corresponds to fixing an arbitrary coordinate system, in which $d\mu$ has the density $\rho(x)$, and then changing $L(x, u, du)\rho(x)$ into $L(x, u, du)$. The proof will be the same. With an analogous simplification purpose, we will replace $\exp_u tX$ by $u + tX(u) + o(t)$ in the calculations (for small t). If we choose to use local charts on $\mathcal{N}$, and hence view u as a map from Ω to an open subset of $\mathbb{R}^n$, this will not be a big problem. In the case where we choose to represent $\mathcal{N}$ as a submanifold of $\mathbb{R}^N$, we should extend L, a priori defined over $T\mathcal{N} \otimes_{\mathcal{M}\times\mathcal{N}} T^*\mathcal{M}$, to a $\mathcal{C}^1$ Lagrangian function on $\Omega \times \mathbb{R}^N \times (\mathbb{R}^N \otimes T^*_x\mathcal{M})$. Such a construction does not pose any particular problem. The fact of u being a critical point of $\mathcal{L}$ implies, in particular, that for every test function $\phi \in \mathcal{C}^\infty_c(\Omega, \mathbb{R})$,

$$\begin{aligned} \mathcal{L}(\exp_u(t\phi X)) &= \mathcal{L}(u + t\phi X + o(t)) \\ &= \mathcal{L}(u) + o(t)\,. \end{aligned} \tag{1.31}$$

But

$$\begin{aligned}
&\mathcal{L}(u + t\phi X + o(t)) \\
&\quad = \int_\Omega L(x, u + t\phi X(u), du + t\phi d(X(u)) + td\phi \cdot X(u))dx^1 \ldots dx^m \\
&\qquad + o(t) \\
&\quad = \int_\Omega L(x, u + t\phi X(u), du + t\phi d(X(u)))dx^1 \ldots dx^m \\
&\qquad + t \int_\Omega X^i \frac{\partial \phi}{\partial x^\alpha} \frac{\partial L}{\partial A^i_\alpha}(x, u, du)dx^1 \ldots dx^m + o(t)\,.
\end{aligned}$$

By relation (1.28),

$$L(x, u + t\phi X(u), du + t\phi d(X(u))) = L(x, u, du) + o(t),$$

and so

$$\mathcal{L}(u+t\phi X+o(t)) = \mathcal{L}(u)+\int_\Omega (X^i \frac{\partial L}{\partial A^i_\alpha}(x, u, du))\frac{\partial \phi}{\partial x^\alpha} dx^1 \ldots dx^m + o(t)\,. \tag{1.32}$$

Comparing (1.31) and (1.32), we obtain

$$\int_\Omega \frac{\partial \phi}{\partial x^\alpha}(X^i \frac{\partial L}{\partial A^i_\alpha}(x, u, du))dx^1 \ldots dx^m = 0, \quad \forall \phi \in \mathcal{C}^\infty_c(\Omega, \mathbb{R})\,,$$

which is the variational formulation of (1.30). □

As an example of applying this result, let us consider the case of harmonic maps. We have

$$\begin{aligned}
L(x, y, A) &= \tfrac{1}{2} g^{\alpha\beta}(x) h_{ij}(y) A^i_\alpha A^j_\beta \\
&= \tfrac{1}{2} g^{\alpha\beta}(x) \langle A_\alpha, A_\beta \rangle\,,
\end{aligned}$$

and

$$\rho(x) = \sqrt{\det g}\, dx^1 \ldots dx^m\,.$$

Then X is an infinitesimal symmetry if the flow generated by X is a family of isometries of $(\mathcal{N}, h)$, i.e. if X is a Killing vector field. Such fields are characterized by the fact that

$$L_X h = 0,$$

where L is the Lie derivative. In local coordinates, this is written as

$$
\begin{aligned}
(L_X h)_{ij} &= X^k \frac{\partial h_{ij}}{\partial y^k} + \frac{\partial X^k}{\partial y^i} h_{kj} + \frac{\partial X^k}{\partial y^j} h_{ik} \\
&= h_{jk} \nabla_{\frac{\partial}{\partial y^i}} X^k + h_{ik} \nabla_{\frac{\partial}{\partial y^j}} X^k = 0 \,.
\end{aligned}
$$

The Noether current is then the vector field

$$
J = J^\alpha \frac{\partial}{\partial x^\alpha} = g^{\alpha\beta}(x) \left\langle X(u), \frac{\partial u}{\partial x^\beta} \right\rangle \sqrt{\det g}\, \frac{\partial}{\partial x^\alpha} \,.
$$

Example 1.3.3 *Let us come back to the sphere – this time of arbitrary dimension n:*

$$
\mathcal{N} = S^n = \{ y \in \mathbb{R}^{n+1} \mid |y| = 1 \} \,.
$$

The group of rotations of $\mathbb{R}^{n+1}$, $SO(n+1)$, leaves S^n invariant, and acts isometrically on S^n. The set of Killing vector fields on S^n can be identified with the Lie algebra $so(n+1)$ of $SO(n+1)$. We will systematically identify $so(n+1)$ with the set of $(n+1)\times(n+1)$ skew-symmetric real matrices, and

$$
SO(n+1) = \{ R \in M((n+1)\times(n+1), \mathbb{R}) \mid {}^t R R = \mathbb{1}, det\, R = 1 \}.
$$

In fact, to each element $a \in so(n+1)$, we may associate the tangent vector field on S^n given by

$$
X_a : y \longmapsto a.y \,,
$$

and we can obtain all Killing vector fields in this way. The previous theorem states that the Noether current

$$
J = J^\alpha \frac{\partial}{\partial x^\alpha} = \left\langle a.u, \frac{\partial u}{\partial x^\beta} \right\rangle g^{\alpha\beta}(x) \sqrt{det\, g}\, \frac{\partial}{\partial x^\alpha} \tag{1.33}
$$

is divergence-free, i.e.

$$
\sum_{\alpha=1}^m \frac{\partial J^\alpha}{\partial x^\alpha} = \sum_{\alpha=1}^m \frac{\partial}{\partial x^\alpha} \left(g^{\alpha\beta}(x) \left\langle a.u, \frac{\partial u}{\partial x^\beta} \right\rangle \sqrt{det\, g} \right) = 0 \,.
$$

Example 1.3.4 *(continuation of the previous example). We restrict ourselves to the case where Ω is an open subset of $\mathbb{R}^m$ equipped with the Euclidean metric, and where $n = 2$. By successively choosing for $a \in so(3)$*

$$
\begin{aligned}
a_1 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \\
a_2 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \\
a_3 &= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\end{aligned}
$$

we obtain the currents (using (1.30))

$$
\begin{aligned}
j_{1,\alpha} &= u^2 \frac{\partial u^3}{\partial x^\alpha} - u^3 \frac{\partial u^2}{\partial x^\alpha}, \\
j_{2,\alpha} &= u^3 \frac{\partial u^1}{\partial x^\alpha} - u^1 \frac{\partial u^3}{\partial x^\alpha}, \\
j_{3,\alpha} &= u^1 \frac{\partial u^2}{\partial x^\alpha} - u^2 \frac{\partial u^1}{\partial x^\alpha}.
\end{aligned}
$$

We recognize that $(j_{1,\alpha}, j_{2,\alpha}, j_{3,\alpha})$ are the three components of the vector $u \times \frac{\partial u}{\partial x^\alpha}$, and Noether's theorem yields

$$\sum_{\alpha=1}^{3} \frac{\partial}{\partial x^\alpha} \left(u \times \frac{\partial u}{\partial x^\alpha} \right) = 0.$$

Thus, we recover (1.26).

This equation is particularly interesting in the case of a sphere (also for other homogeneous manifolds), since the isometry group $SO(n+1)$ acts transitively on S^n and, at an infinitesimal scale, this yields that at each point y of S^n, the set of all $a.y$, for $a \in so(n+1)$, is equal to $T_y S^n$ (equivalently if a_i is a basis of $so(n+1)$, then the $a_i.y$'s span $T_y S^n$). This implies that equation (1.26) is equivalent to the original S^2-valued harmonic map equation, (1.24). We will take advantage of this in sections 2.5 and 2.6.

1.3.2 Symmetries on $\mathcal{M}$: the stress–energy tensor

Let us start with an example: when the manifold $\mathcal{M}$ is 2-dimensional, surprising conservation laws occur. Let us place ourselves in local conformal coordinates (x, y) on $\mathcal{M}$, and consider the $\mathbb{C}$-valued function defined by

$$f = \left|\frac{\partial u}{\partial x}\right|^2 - \left|\frac{\partial u}{\partial y}\right|^2 - 2i\left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right\rangle .$$

One can check that when u is a harmonic map taking values in a Riemannian manifold $(\mathcal{N}, h)$, f is a holomorphic function of the variable $z = x + iy$. This was first noticed, in some special cases, by H.E. Rauch and K. Shibata, then by J. Sampson ([143]), in the 1960s, before people realized around 1980 that this result is general and related (through Noether's theorem) to the problem's invariance under conformal transformations ([5], [127]).

Remark 1.3.5 *A geometric characterization of f is the following: the inverse image (or pull-back) by u of the metric on $\mathcal{N}$,*

$$u^*h = |u_x|^2(dx)^2 + \langle u_x, u_y\rangle (dx \otimes dy + dy \otimes dx) + |u_y|^2(dy)^2 ,$$

may be decomposed over the complexified tangent space $T_x\mathcal{M} \otimes \mathbb{C}$, as

$$u^*h = \frac{1}{4}[f(dz)^2 + |du|^2(dz \otimes d\overline{z} + d\overline{z} \otimes dz) + \overline{f}(d\overline{z})^2] .$$

This decomposition does not depend on the local (conformal) coordinate system chosen. Thus, we can define f by the relation $4(u^*h)^{(2,0)} = f(dz)^2$.

We will come back to this in definition 1.3.10. First we will study the general case where $(\mathcal{M}, g)$ is an arbitrary manifold of any dimension.

Let Φ be a diffeomorphism of the open subset $\Omega \subset (\mathcal{M}, g)$ onto itself, and u be a map from Ω to $\mathcal{N}$. The pull-back of g by Φ is the tensor Φ^*g defined by

$$(\Phi^*g)_{\alpha\beta} = g_{\gamma\delta}(\Phi(x))\frac{\partial \Phi^\gamma}{\partial x^\alpha}\frac{\partial \Phi^\delta}{\partial x^\beta}. \tag{1.34}$$

Φ^*g defines a new metric over Ω, such that (Ω, Φ^*g) and (Ω, g) are "geometrically identical" manifolds, in the sense that

$$\Phi : (\Omega, \Phi^*g) \longrightarrow (\Omega, g)$$

is an isometry. This is why we consider that the two maps

$$u : (\Omega, g) \longrightarrow (\mathcal{N}, h)$$

and

$$u \circ \Phi : (\Omega, \Phi^* g) \longrightarrow (\mathcal{N}, h)$$

are two different representations of a same geometric object. Moreover, the transformation of u into $u \circ \Phi$, which is just a change of coordinates, preserves the Dirichlet integral, provided that we also transform the metric. In fact, a change of variable,

$$\phi = \Phi(x)\,,$$

yields

$$\begin{aligned} E_{(\Omega,\Phi^* g)}(u \circ \Phi) &= \int_\Omega g^{\gamma\delta}(\Phi(x)) \frac{\partial x^\alpha}{\partial \Phi^\gamma} \frac{\partial x^\beta}{\partial \Phi^\delta} \left\langle \frac{\partial (u \circ \Phi)}{\partial x^\alpha}, \frac{\partial (u \circ \Phi)}{\partial x^\beta} \right\rangle \\ &\quad \sqrt{\det \Phi^* g}\, dx^1 \ldots dx^m \\ &= \int_\Omega g^{\gamma\delta}(\phi) \left\langle \frac{\partial u}{\partial \phi^\alpha}, \frac{\partial u}{\partial \phi^\beta} \right\rangle \sqrt{\det g(\phi)}\, d\phi^1 \ldots d\phi^m \\ &= E_{(\Omega,g)}(u). \end{aligned}$$

Hence, we notice the invariance of the Dirichlet integral under the action of the diffeomorphism group

$$\mathrm{Diff}(\Omega) = \{\Phi \in \mathcal{C}^1(\Omega,\Omega) \,|\, \Phi \text{ is invertible and } \ \Phi^{-1} \in \mathcal{C}^1(\Omega,\Omega)\}$$

on the pairs (g, u). We expect to find conservation laws. Nevertheless, two remarkable differences should be pointed out, when compared to the classical setting of Noether's theorem (like the one we saw before). First, the group $\mathrm{Diff}(\Omega)$ is infinite dimensional. Next, the Lagrangian of the harmonic map is not exactly invariant under this group action, to the extent that we need to change the metric g, and thus the Lagrangian L, at the same time as we change the map u, in order to obtain the invariance. The result we will obtain will not concern the existence of divergence-free vector fields, but that of an order 2 symmetric tensor, whose covariant derivative (w.r.t. the metric g) will vanish. Such a tensor S will be defined by

$$S = e(u) g - u^* h\,, \tag{1.35}$$

where $e(u)$ is the energy density. In local coordinates

$$\begin{aligned} S_{\alpha\beta}(x) &= \frac{1}{2}h_{ij}(u(x))g^{\gamma\delta}(x)\frac{\partial u^i}{\partial x^\gamma}\frac{\partial u^j}{\partial x^\delta}g_{\alpha\beta} - h_{ij}(u(x))\frac{\partial u^i}{\partial x^\alpha}\frac{\partial u^j}{\partial x^\beta} \\ &= \frac{1}{2}g^{\gamma\delta}(x)\left\langle \frac{\partial u}{\partial x^\gamma}, \frac{\partial u}{\partial x^\delta}\right\rangle g_{\alpha\beta}(x) - \left\langle \frac{\partial u}{\partial x^\alpha}, \frac{\partial u}{\partial x^\beta}\right\rangle . \end{aligned}$$

The tensor S is called the stress–energy tensor (this name comes from general relativity where its variational origin was first pointed out by David Hilbert [91]). The following result is due to Paul Baird, James Eells and A.I. Pluzhnikov.

Theorem 1.3.6 *[5], [127] Let $u : (\mathcal{M}, g) \longrightarrow (\mathcal{N}, h)$ be a smooth harmonic map. Then, the stress–energy tensor of u, S, has vanishing covariant divergence. This can be stated in two equivalent ways:*

(i) *For every compactly supported tangent vector field X on $\mathcal{M}$,*

$$\int_{\mathcal{M}} (L_X g^\sharp)^{\alpha\beta} S_{\alpha\beta}\, \mathrm{dvol}_g = 0\,, \tag{1.36}$$

where

$$(L_X g^\sharp)^{\alpha\beta} = X^\gamma \frac{\partial g^{\alpha\beta}}{\partial x^\gamma} - \frac{\partial X^\alpha}{\partial x^\gamma} g^{\gamma\beta} - \frac{\partial X^\beta}{\partial x^\gamma} g^{\alpha\gamma}$$

is the Lie derivative of $g^\sharp$ with respect to X.

(ii) *The tensor S satisfies the following equation for $\beta = 1, \ldots, m$,*

$$\mathrm{div}_g(g^{\alpha\beta} S_{\gamma\beta}) = \nabla_{\frac{\partial}{\partial x^\alpha}}(g^{\alpha\gamma} S_{\gamma\beta}) = 0\,. \tag{1.37}$$

Proof We first prove (i). Let X be a compactly supported Lipschitz vector field on $\mathcal{M}$, and Φ_t the flow generated by X,

$$\left\{ \begin{array}{l} \Phi_0(x) = x, \ \forall x \in \mathcal{M} \\ \dfrac{\partial(\Phi_t(x))}{\partial t} = X(\Phi_t(x))\,. \end{array} \right.$$

For all t, Φ_t exists and is an element of Diff($\mathcal{M}$).