CAMBRIDGE TRACTS IN MATHEMATICS

150 Harmonic maps, conservation laws and moving frames

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to Henry Wente

Contents

Foreword

Harmonic maps between Riemannian manifolds provide a rich display of both differential geometric and analytic phenomena. These aspects are inextricably intertwined — a source of undiminishing fascination.

Analytically, the problems belong to elliptic variational theory: *harmonic maps* are the solutions of the Euler–Lagrange equation (section 1.2)

$$\Delta_g u^i + g^{\alpha\beta}(x)\Gamma^i_{jk}(u(x))\frac{\partial u^j}{\partial x^\alpha}\frac{\partial u^k}{\partial x^\beta} = 0 \qquad (1)$$

associated to the Dirichlet integral (section 1.1)

$$E(u) = \int_{\mathcal{M}} \frac{|du(x)|^2}{2} d\text{vol}_g\,.$$

Surely that is amongst the simplest — and yet general — intrinsic variational problems of Riemannian geometry. The system (1) is second order elliptic of divergence type, with linear principal parts in diagonal form with the same Laplacian in each entry; and whose first derivatives have quadratic growth. That is quite a restrictive situation; indeed, those conditions ensure the regularity of continuous weak solutions of (1).

The entire harmonic mapping scene (as of 1988) is surveyed in the articles [50] and [51].

2-dimensional domains

Harmonic maps $u : \mathcal{M} \longrightarrow \mathcal{N}$ with 2-dimensional domains $\mathcal{M}$ present special features, crucial to their applications to minimal surfaces (i.e. conformal harmonic maps) and to deformation theory of Riemann surfaces. Amongst these, as they appear in this monograph:

(i) The Dirichlet integral is a conformal invariant of $\mathcal{M}$. Consequently, harmonicity of u (characterized via the Euler–Lagrange operator associated to E) depends only on the conformal structure of $\mathcal{M}$ (section 1.1).
(ii) Associated with a harmonic map is a holomorphic quadratic differential on $\mathcal{M}$ (locally represented by the function f of section 1.3).
(iii) The inequality of Wente. Qualitatively, that ensures that the Jacobian determinant of a map u (a special quadratic expression involving first derivatives of u) may have slightly more differentiability than might be expected (section 3.1).
(iv) The $\mathcal{C}^2$ maps are dense in $H^1(\mathcal{M}, \mathcal{N})$†

To gain a perspective on the use of harmonic maps of surfaces, the reader is advised to consult [48] and [116] for minimal surfaces and the problem of Plateau. Applications to the theory of deformations of Riemann surfaces can be found in [68] and [49]. The book [98] provides an introduction to all these questions.

Regularity

A key step in Morrey's solution of the Plateau problem is his

Theorem 1 (Morrey) *Let $\mathcal{M}$ be a Riemann surface, and $u : \mathcal{M} \longrightarrow \mathcal{N}$ a map with $E(u) < +\infty$. Suppose that u minimizes the Dirichlet integral E_B on every disk B of $\mathcal{M}$ (with respect to the Dirichlet problem induced by the trace of u on the boundary of B). Then u is Hölder continuous.*

In particular, u is harmonic (and as regular as the data permits).

The proof is based on Morrey's Dirichlet growth estimate — related to the growth estimates in section 3.5.

The main goal of the present monograph is the following result, giving a definitive generalization of Theorem 1:

Theorem 2 (Hélein) *Let $(\mathcal{M}, g)$ be a Riemann surface, and $(\mathcal{N}, h)$ a compact Riemannian manifold without boundary. If $u : \mathcal{M} \longrightarrow \mathcal{N}$ is a weakly harmonic map with $E(u) < +\infty$, then u is harmonic.*

† See the proof of lemma 4.1.6 and [145].

That is indeed a major achievement, made some fifty years after Morrey's special case. Hélein first established his theorem in certain particular cases ($\mathcal{N} = S^n$ and various Riemannian homogeneous spaces); then he announced Theorem 2 in [85]. That Note includes a beautifully clear sketch of the proof, together with a description of the new ideas — an absolute gem of presentation!

The high quality is maintained here:

Commentary on the text

First of all, the author's exposition requires only a few formalities from differential geometry and variational theory. Secondly, the pace is leisurely and well motivated throughout.

For instance: chapter 1 develops the required background for harmonic maps. The author is satisfied with maps and Riemannian metrics of differentiability class $\mathcal{C}^2$; higher differentiability then follows from general principles. Various standard conservation laws are derived. All that is direct and efficient.

As a change of scene, chapter 2 is an excursion into the methods of completely integrable systems, as applied to harmonic maps of a Riemann surface into S^n (or a Lie group; or a homogeneous space), via conservation laws. One purpose is to illustrate hidden symmetries of Lax form (e.g. related to dressing action). Another is to provide motivations for the methods and constructions used in chapter 4 — especially the role of symmetry in the range.

Chapter 3 describes various spaces of functions — Hardy and Lorentz spaces, in particular — as an exposition specially designed for applications in chapters 4 and 5. Those include refinements and modifications of Wente's inequality; and come under the heading of compensation phenomena — certainly delicate and lovely mathematics!

Chapter 4 is the heart of the monograph — as already noted. There are two new steps required as preparation for the proof of theorem 2:

(i) Lemma 4.1.2, which reduces the problem to the case in which $(\mathcal{N}, h)$ is a Riemannian manifold diffeomorphic to a torus.

(ii) Careful construction of a special frame field on $(\mathcal{N}, h)$ — called a Coulomb frame. Equations (4.10) are derived, serving as some sort of conservation law. When the spaces of Hardy and Lorentz enter the scene, they produce a gain of regularity (see lemma 4.1.7).

Finally, in chapter 5 the methods of Coulomb frames and compensation techniques are applied to problems of surfaces in Euclidean spaces whose second fundamental form or mean curvatures are square-integrable.

James Eells

Introduction

The contemplation of the atlas of an airline company always offers us something puzzling: the trajectories of the airplanes look curved, which goes against our basic intuition, according to which the shortest path is a straight line. One of the reasons for this paradox is nothing but a simple geometrical fact: on the one hand our earth is round and on the other hand the shortest path on a sphere is an arc of great circle: a curve whose projection on a geographical map rarely coincides with a straight line. Actually, choosing the trajectories of airplanes is a simple illustration of a classical variational problem in differential geometry: finding the geodesic curves on a surface, namely paths on this surface with minimal lengths.

Using water and soap we can experiment an analogous situation, but where the former path is now replaced by a soap film, and for the surface of the earth — which was the ambient space for the above example — we substitute our 3-dimensional space. Indeed we can think of the soap film as an excellent approximation of some ideal elastic matter, infinitely extensible, and whose equilibrium position (the one with lowest energy) would be either to shrink to one point or to cover the least area. Thus such a film adopts a minimizing position: it does not minimize the length but the area of the surface. Here is another classical variational problem, the study of minimal surfaces.

Now let us try to imagine a 3-dimensional matter with analogous properties. We can stretch it inside any geometrical manifold, as for instance a sphere: although our 3-dimensional body will be confined — since generically lines will shrink to points — it may find an equilibrium configuration. Actually the mathematical description of such a situation, which is apparently more abstract than the previous ones, looks like the mathematical description of a nematic liquid crystal in equilibrium.

Such a bulk is made of thin rod shaped molecules (*nema* means thread in Greek) which try to be parallel each to each other. Physicists have proposed different models for these liquid crystals where the mean orientation of molecules around a point in space is represented by a vector of norm 1 (hence some point on the sphere). Thus we can describe the configuration of the material using a map defined in the domain filled by the liquid crystal, with values into the sphere. We get a situation which is mathematically analogous to the abstract experiment described above, by imagining we are trying to imprison a piece of perfectly elastic matter inside the surface of a sphere. The physicists Oseen and Frank proposed a functional on the set of maps from the domain filled with the material into the sphere, which is very close to the elastic energy of the abstract ideal matter.

What makes all these examples similar (an airplane, water with soap and a liquid crystal)? We may first observe that these three situations illustrate variational problems. But the analogy is deeper because each of these examples may be modelled by a map (describing the deformation of some body inside another one) which maps a differential manifold into another one, and which minimizes a quantity which is more or less close to a perfect elastic energy. To define that energy, we need to measure the infinitesimal stretching imposed by the mapping and to define a measure on the source space. Such definitions make sense provided that we use Riemannian metrics on the source and target manifolds.

Let $\mathcal{M}$ denote the source manifold, $\mathcal{N}$ the target manifold and u a differentiable map from $\mathcal{M}$ into $\mathcal{N}$. Given Riemannian metrics on these manifolds we may define the *energy* or *Dirichlet integral*

$$E(u) = \frac{1}{2}\int_{\mathcal{M}} |du|^2 d\text{vol},$$

where $|du|$ is the Hilbert–Schmidt norm of the differential du of u and $d$vol is the Riemannian measure on $\mathcal{M}$. If we think of the map u as the way to confine and stretch an elastic $\mathcal{M}$ inside a rigid $\mathcal{N}$, then $E(u)$ represents an elastic deformation energy. Smooth maps (i.e. of class $\mathcal{C}^2$) which are critical points of the Dirichlet functional are called *harmonic maps*. For the sake of simplicity, let us assume that $\mathcal{N}$ is a submanifold of a Euclidean space. Then the equation satisfied by a harmonic map is

$$\Delta u(x) \perp T_{u(x)}\mathcal{N},$$

where Δ is the Laplacian on $\mathcal{M}$ associated to the Riemannian metric, and $T_{u(x)}\mathcal{N}$ is the tangent space to $\mathcal{N}$ at the point $u(x)$. For different choices on $\mathcal{M}$ and $\mathcal{N}$, a harmonic map will be a constant speed parametrization of a geodesic (if the dimension of $\mathcal{M}$ is 1), a harmonic function (if $\mathcal{N}$ is the real line) or something hybrid.

It is possible to extend the notion of harmonic maps to much less regular maps, which belong to the Sobolev space $H^1(\mathcal{M}, \mathcal{N})$ of maps from $\mathcal{M}$ into $\mathcal{N}$ with finite energy. The above equation is true but only in the distribution sense and we speak of *weakly harmonic maps*.

Because of the simplicity of this definition, we can meet examples of harmonic maps in various situations in geometry as well as in physics. For example, any submanifold $\mathcal{M}$ of an affine Euclidean space has a constant mean curvature (or more generally a parallel mean curvature) if and only if its Gauss map is a harmonic map. A submanifold $\mathcal{M}$ of a manifold $\mathcal{N}$ is minimal if and only if the immersion of $\mathcal{M}$ in $\mathcal{N}$ is harmonic. In condensed matter physics, harmonic maps between a 3-dimensional domain and a sphere have been used as a simplified model for nematic liquid crystals. In theoretical physics, harmonic maps between surfaces and Lie groups are extensively studied, since they lead to properties which are strongly analogous to (anti)self-dual Yang–Mills connections on 4-dimensional manifolds, but they are simpler to handle. In such a context they correspond to the so-called σ-models. Recently, the interest of physicists in these objects has been reinforced since their quantization leads to examples of conformal quantum field theories — an extremely rich subject. In some sense the quantum theory for harmonic maps between a surface and an Einstein manifold (both endowed with Minkowski metrics) corresponds to string theory (in the absence of supersymmetries). Other models used in physics, such as the Skyrme model, Higgs models or Ginzburg–Landau models [12], show strong connections with the theory of harmonic maps into a sphere or a Lie group.

Despite their relatively universal character, harmonic maps became an active topic for mathematicians only about four decades ago. One of the first questions was motivated by algebraic topology: given two Riemannian manifolds and a homotopy class for maps between these manifolds, does there exist a harmonic map in this homotopy class? In the case where the sectional curvature of the target manifold is negative, James Eells and Joseph Sampson showed in 1964 that this is true, using the heat equation. Then the subject developed in many different directions and aroused many fascinating questions in topology, in differential geometry, in algebraic geometry and in the analysis of partial differential equations. Important generalizations have been proposed, such as the evolution equations for harmonic maps between manifolds (heat equation or wave equation) or the p-harmonic maps (i.e. the critical points of the integral of the p-th power of $|du|$). During the same period, and essentially independently, physicists also developed many interesting ideas on the subject.

The present work does not pretend to be a complete presentation of the theory of harmonic maps. My goal is rather to offer the reader an introduction to this subject, followed by a communication of some recent results. We will be motivated by some fundamental questions in analysis, such as the compactness in the weak topology of the set of weakly harmonic maps, or their regularity. This is an opportunity to explore some ideas and methods (symmetries, compensation phenomena, the use of *moving frames* and of *Coulomb moving frames*), the scope of which is, I believe, more general than the framework of harmonic maps.

The regularity problem is the following: is a weakly harmonic map u smooth? (for instance if $\mathcal{N}$ is of class $\mathcal{C}^{k,\alpha}$, is u of class $\mathcal{C}^{k,\alpha}$, for $k \geq 2$, $0 < \alpha < 1$?). The (already) classical theory of quasilinear elliptic partial differential equation systems ([117], [103]) teaches us that any *continuous* weakly harmonic map is automatically as regular as allowed by the the regularity of the Riemannian manifolds involved. The critical step is thus to know whether or not a weakly harmonic map is continuous. Answers are extremely different according to the dimension of the source manifold, the curvature of the target manifold, its topology or the type of definition chosen for a weak solution.

The question of compactness in the weak topology of weakly harmonic maps is the following problem. Given a sequence $(u_k)_{k\in\mathbb{N}}$ of weakly harmonic maps which converge in the weak topology of $H^1(\mathcal{M},\mathcal{N})$ towards a map u, can we deduce that the limit u is a weakly harmonic map? Such a question arises when, for instance, one wants to prove the existence of solutions to evolution problems for maps between manifolds. This is a very disturbing problem: we will see that the answer is yes in the case where the target manifold is symmetric, but we do not know the answer in the general situation.

The first idea which this book stresses is the role of symmetries in a variational problem. It is based on the following observation, due to Emmy Noether: if a variational problem is invariant under the action of a continuous group of symmetries, we can associate to each solution of this variational problem a system of conservation laws, i.e. one, or several, divergence-free vector fields defined on the source domain. The number of independent conservation laws is equal to the dimension of the group of symmetries. The importance of this result has been celebrated for years in theoretical physics. For example, in the particular case where the variational problem involves one variable (the time) the conservation law is just the prediction that a scalar quantity is constant in time (the conservation of the energy comes from the invariance under time translations, the conservation of the momentum is a consequence of the invariance under translations in space...). One of the goals of this book is to convince you that Noether's theorem is also fundamental in the study of partial differential equations such as harmonic maps.

In a surprising way, the exploitation of symmetries for analytical purposes is strongly related to compensation phenomena: by handling conservation laws, remarkable non-linear quantities (for an analyst) naturally appear. The archetype of this kind of quantity is the Jacobian determinant

$$\{a, b\} := \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x},$$

where a and b are two functions whose derivatives are square-integrable (i.e. a and b belong to the Sobolev space H^1). Since Charles B. Morrey, it is known that such a quantity enjoys the miraculous property of being simultaneously non-linear and continuous with respect to the weak topology in the space H^1; if a_ϵ and b_ϵ converge *weakly* in H^1 towards a and b, then $\{a_\epsilon, b_\epsilon\}$ converges towards $\{a, b\}$ in the sense of distributions. This is the subject of the theory of compensated compactness of François Murat and Luc Tartar. Moreover the same quantity $\{a, b\}$ possesses regularity or integrability properties slightly better than any other bilinear function of the partial derivatives of a and b. It seems that this result of "compensated regularity" was observed for the first time by Henry Wente in 1969. For twenty years this phenomenon was used only in the context of constant mean curvature surfaces, by H. Wente and by Haïm Brezis and Jean-Michel Coron (further properties were pointed out by these last two authors and also by L. Tartar). But more recently, at the end of the 1980s, works of Stefan Müller, followed by Ronald Coifman, Pierre-Louis Lions, Yves Meyer and Stephen Semmes, shed a new light on the quantity $\{a, b\}$, and in particular it was established that this Jacobian determinant belongs to the *Hardy space*, a slightly improved version of the space of integrable functions L^1. All these results played a vital role in the progress which has been obtained recently in the regularity theory of harmonic maps, and are the companion ingredients to the conservation laws.

The limitation of techniques which use conservation laws is that symmetric variational problems are exceptions. Thus the above methods are not useful, a priori, for the study of harmonic maps with values into a non-symmetric manifold. We need then to develop new techniques. One idea is the use of moving frames. It consists in giving, for each point y in $\mathcal{N}$, an orthonormal basis $(e_1, \ldots, e_n)$ of the tangent space to $\mathcal{N}$ at y, that depends smoothly on the point y. This system of coordinates on the tangent space $T_y\mathcal{N}$ was first developed by Gaston Darboux and mainly by Elie Cartan. These moving frames turn out to be extremely suitable in differential geometry and allow a particularly elegant presentation of the Riemannian geometry (see [37]). But in the problems with which we are concerned, we will use a particular class of moving frames, satisfying an extra differential equation. It consists essentially of a condition which expresses that the moving frame is a harmonic section (a generalization of harmonic maps to the case of fiber bundles) of a fiber bundle over $\mathcal{M}$ whose fiber at x is precisely the set of orthonormal bases of the tangent space to $\mathcal{N}$ at $u(x)$. Since the rotation group $SO(n)$ is a symmetry group for that bundle and for the associated variational problem, our condition gives rise to conservation laws, thanks to Noether's theorem. We call such a moving frame a *Coulomb moving frame*, inspired by the analogy with the use of Coulomb gauges by physicists for gauge theories. The use of such a system of privileged coordinates is crucial for the study of the regularity of weakly harmonic maps, with values into an arbitrary manifold. It leads to the appearance of these magical quantities similar to $\{a, b\}$, that we spoke about before.

The first chapter of this book presents a description of harmonic maps and of various notions of weak solutions. We will emphasize Noether's theorem through two versions which play an important role for harmonic maps. In the (exceptional but important) case where the target manifold $\mathcal{N}$ possesses symmetries, the conservation laws lead to very particular properties which will be presented in the second chapter. But in constrast, there is a symmetry which is observed in general cases and which is related to invariance under change of coordinates on the source manifold $\mathcal{M}$. It is not really a geometrical symmetry in general and it will lead to some covariant version of Noether's theorem: the stress–energy tensor

$$S_{\alpha\beta} = \frac{|du|^2}{2} g_{\alpha\beta} - \left\langle \frac{\partial u}{\partial x^\alpha}, \frac{\partial u}{\partial x^\beta} \right\rangle$$

always has a vanishing covariant divergence. This equation has a consequence which is very important for the theory of the regularity of weak solutions: the monotonicity formula. In the case where there is a geometrical symmetry acting on $\mathcal{M}$, some of the covariant conservation laws specialize and become true conservation laws. One particular case is when the dimension of $\mathcal{M}$ is 2, since then the harmonic map problem is invariant under conformal transformations of $\mathcal{M}$, and hence the stress–energy tensor coincides with the *Hopf differential* and is holomorphic. We end this chapter by a quick survey of the regularity results which are known concerning weak solutions.

The second chapter is a suite of variations on the version of Noether's theorem which concerns harmonic maps with values into a symmetric manifold $\mathcal{N}$. We present various kinds of results but they are all consequences of the same conservation law. If for instance $\mathcal{N}$ is the sphere S^2 in 3-dimensional space, we start from

$$\operatorname{div}(u^i \nabla u^j - u^j \nabla u^i) = 0, \ \forall i, j = 1, 2, 3.$$

Using this conservation law, we will see that it is easy to exhibit the relations between harmonic maps from a surface into S^2, and surfaces of constant mean curvature or positive constant Gauss curvature in 3-dimensional space. We hence recover the construction due to Ossian Bonnet of families of parallel surfaces with constant mean curvature and constant Gauss curvature. Moreover, we can deduce from this conservation law a formulation (which was probably discovered by K. Pohlmeyer and by V.E. Zhakarov and A.B. Shabat) using loop groups, of the harmonic maps problem between a surface and a symmetric manifold. Such a formulation is a feature of completely integrable systems, like the Korteweg–de Vries equation (see [150]). Many authors have used this theory during the last decade in a spectacular way: Karen Uhlenbeck deduced a classification of all harmonic maps from the sphere S^2 into the group $U(n)$ [174]. After Nigel Hitchin, who obtained all harmonic maps from a torus into S^3 by algebraico-geometric methods [94], Fran Burstall, Dirk Ferus, Franz Pedit and Ulrich Pinkall were able to construct all harmonic maps from a torus into a symmetric manifold (the symmetry group of which is compact semi-simple) [24] and more recently an even more general construction has been obtained by Joseph

Dorfmeister, Franz Pedit and HongYu Wu [46]. We will give a brief description of some of these results.

In another direction, the same conservation law allows one to prove in a few lines some analysis results such as the compactness in the weak topology of the set of weakly harmonic maps with values into a symmetric manifold, or their regularity (complete or partial depending on other hypotheses): we present the existence result for solutions to the wave equation for maps with values in a symmetric manifold due to Jalal Shatah, and my regularity result for weakly harmonic maps between a surface and a sphere.

The third chapter, which is essentially devoted to compensation phenomena and to Hardy and Lorentz spaces, brings very different ingredients by constrast with the previous chapter, but complementary. The main object is the Jacobian determinant $\{a, b\}$. We begin by showing the following result due to H. Wente: if a and b belong to the Sobolev space $H^1(\Omega, \mathbb{R})$, where Ω is a domain in the plane $\mathbb{R}^2$, and if ϕ is the solution on Ω of

$$\begin{cases} -\Delta\phi & = & \{a, b\} & \text{in } \Omega \\ \phi & = & 0 & \text{on } \partial\Omega, \end{cases}$$

then ϕ is continuous and is in $H^1(\Omega, \mathbb{R})$. Moreover, we can estimate the norm of ϕ in the spaces involved as a function of the norms of da and db in L^2. Then we will discuss some optimal versions of this theorem and its relations with the isoperimetric inequality and constant mean curvature surfaces. Afterwards we will introduce Hardy and Lorentz spaces and see how they can be used to refine Wente's theorem. As an illustration of these ideas, the chapter ends with the proof of a result of Lawrence Craig Evans on the partial regularity of weakly stationary maps with values into the sphere.

The fourth chapter deals with harmonic maps with values into manifolds without symmetry. We thus need to work without the conservation laws which were at the origin of the results of chapter 2. For the regularity problem we substitute for the conservation laws the use of *Coulomb moving frames* on the target manifold $\mathcal{N}$. Given a map u from $\mathcal{M}$ into $\mathcal{N}$, a Coulomb moving frame consists in an orthonormal frame field on $\mathcal{M}$ which is a *harmonic section* of the pull-back by u of the orthonormal tangent frame bundle on $\mathcal{N}$ (i.e. the fiber bundle whose base manifold is $\mathcal{M}$, obtained by attaching to each point x in $\mathcal{M}$ the set of (direct) orthonormal bases of the tangent space to $\mathcal{N}$ at $u(x)$). Using to this construction and the analytical tools introduced in chapter 3, we may extend the regularity results obtained in the two previous chapters, by dropping the symmetry hypothesis on $\mathcal{N}$: we prove my theorem on the regularity of weakly harmonic maps on a surface, then a generalization of it due to Philippe Choné and lastly the generalization of the result of L.C. Evans proved in chapter 3, obtained by Fabrice Bethuel. Strangely, we are not able to present a definite answer to the compactness problem in the weak topology of the set of weakly harmonic maps. Motivated by this question we end chapter 4 by studying the possibility of building conservation laws without symmetries. It leads us to "isometric embedding" problems for covariantly closed differential forms, with coefficients in a vector bundle equipped with a connection. Such problems look interesting by themselves, as this class of questions offers a hybrid generalization of Poincaré's lemma for closed differential forms, and the isometric embedding problem for Riemannian manifolds.

The fifth chapter does not directly concern harmonic maps, but is an excursion into the study of conformal parametrizations of surfaces. The starting point is a result of Tatiana Toro which established the remarkable fact that an embedded surface in Euclidean space which has a square-integrable second fundamental form is Lipschitz. Soon after, Stefan Müller and Vladimir Šverák proved that any conformal parametrization of such a surface is bilipschitz. Their proof relies in a clever way on the compensation results described in chapter 3 about the quantity $\{a, b\}$, and on the use of Hardy space. We give here a slightly different presentation of the result and of the proof of their result: we do not use Hardy space but only Wente's inequality and Coulomb moving frames. More precisely, we study the space of conformal parametrizations of surfaces in Euclidean space with second fundamental form bounded in L^2, and we show a compactness result for this space. This tour will naturally bring us to an amusing interpretation of Coulomb moving frames: a Coulomb moving frame associated to the identity map from a surface to itself corresponds essentially to a system of conformal coordinates.

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Notation

Ω will denote an open subset of $\mathbb{R}^m$.

- $L^p(\Omega)$: Lebesgue space. For $1 \leq p \leq \infty$, $L^p(\Omega)$ is the set (of equivalence classes) of measurable functions f from Ω to $\mathbb{R}$ such that $||f||_{L^p} < +\infty$, where

$$||f||_{L^p} := \left(\int_\Omega |f(x)|^p dx^1 ... dx^m \right)^{\frac{1}{p}}, \text{ if } 1 \leq p < \infty,$$

$$||f||_{L^\infty} := \inf\{M \in [0, +\infty] \mid |f(x)| \leq M \text{ a.e.}\}.$$

- $L^p_{loc}(\Omega)$: space of measurable functions f from Ω to $\mathbb{R}$ such that for every compact subset K of Ω, the restriction of f to K, $f_{|K}$, belongs to $L^p(K)$.
- $W^{k,p}(\Omega)$: Sobolev space. For each multi-index $s = (s_1, ..., s_m) \in \mathbb{N}^m$, we define $|s| = \sum_{\alpha=1}^m s_\alpha$, and $D_s = \frac{\partial^{|s|}}{(\partial x^1)^{s_1} ... (\partial x^m)^{s_m}}$. Then, for $k \in \mathbb{R}$ and $1 \leq p \leq \infty$,

$$W^{k,p}(\Omega) := \{f \in L^p(\Omega) \mid \forall s, |s| \leq k, D_s f \in L^p(\Omega)\}.$$

Here, $D_s f$ is a derivative of order $|s|$ of f, in the sense of distributions. On this space we have the norm

$$||f||_{W^{k,p}} := \sum_{|s| \leq k} ||D_s f||_{L^p}.$$

- $W^{-k,p}(\Omega)$: the dual space of $W^{k,p}(\Omega)$.
- $H^k(\Omega) := W^{k,2}(\Omega)$. On this space we have the norm (equivalent to

$||f||_{W^{k,2}}$)

$$||f||_{H^k} := \left(\sum_{|s| \leq k} ||D_s f||^2_{L^2} \right)^{\frac{1}{2}}.$$

- $\mathcal{C}^k(\Omega)$: set of continuous functions on Ω which are k times differentiable and whose derivatives up to order k are continuous (for $k \in \mathbb{N}$ or $k = \infty$).
- $\mathcal{C}^k_c(\Omega)$: set of functions in $\mathcal{C}^k(\Omega)$ with compact support in Ω.
- $\mathcal{D}'(\Omega)$: space of distributions over Ω (it is the dual of $\mathcal{D}(\Omega) := \mathcal{C}^\infty_c(\Omega)$).
- $\mathcal{C}^{0,\alpha}(\Omega)$: Hölder space. Set of functions f, continuous over Ω, such that

$$\sup_{x,y\in\Omega} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < +\infty,$$

(for $0 < \alpha < 1$).
- $\mathcal{C}^{k,\alpha}(\Omega)$: set of k times differentiable continuous functions such that all derivatives up to order k belong to $\mathcal{C}^{0,\alpha}(\Omega)$.
- $\mathcal{H}^1$: Hardy space, see definitions 3.2.4, 3.2.5 and 3.2.8.
- $BMO(\Omega)$: space of functions with bounded mean oscillation, see definition 3.2.7.
- $L^{(p,q)}(\Omega)$: Lorentz space, see definition 3.3.2.
- $\mathcal{L}^{q,\lambda}(\Omega)$: Morrey–Campanato space, see definition 3.5.9.
- $E_{x,r}$: see example 1.3.7, section 4.3 and section 3.5.
- The scalar product between two vectors X and Y is denoted by $\langle X, Y\rangle$ or $X \cdot Y$.
- $\{a, b\} := \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x}$, see section 3.1.
- $\{u \cdot v\}$: if u and v are two maps from a domain in $\mathbb{R}^2$ with values into a Euclidean vector space $(V, \langle ., .\rangle)$, $\{u \cdot v\} := \langle \frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}\rangle - \langle \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}\rangle$.
- $\underbrace{\{a, b\}}_{\alpha\beta}$: see section 4.3.
- $\widetilde{ab}, \widetilde{ab}_\Omega$: see section 3.1.
- $\Lambda^p\mathbb{R}^m$: algebra of p-forms with constant coefficients over $\mathbb{R}^m$ (p-linear skew-symmetric forms over $\mathbb{R}^m$). $\Lambda\mathbb{R}^m = \bigoplus_{p=0}^m \Lambda^p\mathbb{R}^m$.
- $\wedge$: wedge product in the algebra $\Lambda\mathbb{R}^m$ (see [47] or [183]).
- d: exterior differential, acting linearly over $\mathcal{D}'(\Omega) \otimes \Lambda\mathbb{R}^m$ and such that $\forall\phi \in \mathcal{C}^\infty(\Omega)$, $\forall\alpha \in \Lambda\mathbb{R}^m$, $d(\phi \otimes \alpha) = \sum_{\alpha=1}^m \frac{\partial\phi}{\partial x^\alpha} dx^\alpha \wedge \alpha$.
- $\times$: vector product in $\mathbb{R}^3$:

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \times \begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix} = \begin{pmatrix} x^2y^3 - x^3y^2 \\ x^3y^1 - x^1y^3 \\ x^1y^2 - x^2y^1 \end{pmatrix}.$$

- ${}^t u$: for any vector $u = \begin{pmatrix} u^1 \\ \vdots \\ u^n \end{pmatrix}$, ${}^t u = (u^1, \dots, u^n)$.
- $GL(E)$: if E is a vector space, $GL(E)$ is the group of invertible endomorphisms of E.
- $M(n \times n, \mathbb{R})$ or $M(n \times n, \mathbb{C})$: algebra of (real or complex) square matrices with n rows (and n columns).
- $\mathbb{1}$: identity matrix.
- δ^a_b: Kronecker symbol, its value is 1 if $a = b$ and 0 if $a \neq b$.
- $O(n) := \{R \in M(n \times n, \mathbb{R}) \mid {}^t RR = \mathbb{1}\}$.
- $SO(n) := \{R \in M(n \times n, \mathbb{R}) \mid {}^t RR = \mathbb{1}, \det R = 1\}$.
- $SO(n)^{\mathbb{C}} := \{R \in M(n \times n, \mathbb{C}) \mid {}^t RR = \mathbb{1}, \det R = 1\}$.
- $so(n) := \{A \in M(n \times n, \mathbb{R}) \mid {}^t A + A = 0\}$.
- $SU(n) := \{R \in M(n \times n, \mathbb{C}) \mid {}^t \bar{R} R = \mathbb{1}, \det R = 1\}$.
- $su(n) := \{A \in M(n \times n, \mathbb{C}) \mid {}^t \bar{A} + A = 0, \mathrm{tr} A = 0\}$.
- $Spin(3)$: (2-fold) universal covering of $SO(3)$. It is identified with $SU(2)$.
- $S^{n-1} := \{y \in \mathbb{R}^n \mid |y| = \sqrt{(y^1)^2 + \cdots + (y^n)^2} = 1\}$.