A First Course in Combinatorial Optimization

A First Course in Combinatorial Optimization is a text for a one-semester introductory graduate-level course for students of operations research, mathematics, and computer science. It is a self-contained treatment of the subject, requiring only some mathematical maturity. Topics include linear and integer programming, polytopes, matroids and matroid optimization, shortest paths, and network flows.

Central to the exposition is the polyhedral viewpoint, which is the key principle underlying the successful integer-programming approach to combinatorial-optimization problems. Another key unifying topic is matroids. The author does not dwell on data structures and implementation details, preferring to focus on the key mathematical ideas that lead to useful models and algorithms. Problems and exercises are included throughout as well as references for further study.

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CAMBRIDGE
UNIVERSITY PRESS

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32 Avenue of the Americas, New York NY 10013-2473, USA

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9780521811514

First published 2004
Reprinted 2011

A catalogue record for this publication is available from the British Library

ISBN 978-0-521-81151-4 Hardback
ISBN 978-0-521-01012-2 Paperback

THE HOUSE JACK BUILT

Open doors so I walk inside
Close my eyes find my place to hide
And I shake as I take it in
Let the show begin

Open my eyes
Just to have them close again
Well on my way
On my way to where I graze
It swallows me
As it takes me in his home
I twist away
As I kill this world

Open doors so I walk inside
Close my eyes find my place to hide
And I shake as I take it in
Let the show begin

Open my eyes
Just to have them close once again
Don’t want control
As it takes me down and down again
Is that the moon
Or just a light that lights this deadend street?
Is that you there
Or just another demon that I meet?

The higher you walk
The farther you fall
The longer the walk
The farther you crawl
My body my temple
This temple it tells
“Step into the house that Jack built”

The higher you walk
The farther you fall
The longer the walk
The farther you crawl
My body my temple
This temple it tells
"Yes this is the house that Jack built"

Open doors as I walk inside
Swallow me so the pain subsides
And I shake as I take this in
Let the show begin

The higher you walk
The farther you fall
The longer the walk
The farther you crawl
My body my temple
This temple it tells
"Yes this is the house that Jack built"

The higher you walk
The farther you fall
The longer the walk
The farther you crawl
My body my temple
This temple it tells
"Yes I am I am I am"

Open my eyes
It swallows me
Is that you there
I twist away
Away
Away
Away

– Metallica (Load)

Contents

Preface

This is the house that Jack built. Ralph prepared the lot. There were *many* independent contractors who did beautiful work; some putting on splendid additions. Martin, Laci, and Lex rewired the place. The work continues. But this is the house that Jack built.

This textbook is designed to serve as lecture notes for a one-semester course focusing on combinatorial optimization. I am primarily targeting this at the graduate level, but much of the material may also be suitable for excellent undergraduate students. The goal is to provide an enticing, rigorous introduction to the mathematics of the subject, *within the context of a one-semester course*. There is a strong emphasis on the unifying roles of matroids, submodularity, and polyhedral combinatorics.

I do not pretend that this book is an exhaustive treatment of combinatorial optimization. I do not emphasize data structures, implementation details, or sophisticated approaches that may yield decidedly faster and more practical algorithms. Such are important issues, but I leave them for later independent study. The approach that I take is to focus, mostly, on the beautiful. Also, I note that the terrain of the field shifts rapidly. For example, Gomory's seminal work on integer programming from the 1960s, which was featured prominently in textbooks in the early 1970s, was out of vogue by the late 1970s and through the early 1990s when it was assessed to have no practical value. However, by the late 1990s, Gomory's methods were found to be practically useful. Rather than try and guess as to what will be practically useful some decades from now, I prefer to emphasize some of the work that I regard as foundational.

Also, I do not dwell on applications. To some extent, the applications are the *raison d'être* of combinatorial optimization. However, for the purposes of this book, I take the view that the interesting mathematics and algorithm engineering are justifications enough for studying the subject. Despite (because of?) the fact that I only touch on applications, one can develop talent in modeling and in developing solution methods by working through this book. This apparent paradox is explained by the fact that mathematical abstraction and modeling abstraction are very close cousins.

The prerequisites for studying this book are (1) some mathematical sophistication and (2) *elementary* notions from graph theory (e.g., path, cycle, tree). If one has already studied linear programming, then a good deal of Chapter 0 can be omitted.

Problems (requests for short proofs) and *Exercises* (requests for calculations) are interspersed in the book. Each Problem is designed to teach or reinforce a concept. Exercises are used to either verify understanding of an algorithm or to illustrate an idea. Problems and Exercises should be attempted as they are encountered. I have found it to be very valuable to have students or me present correct solutions to the class on each assignment due date. The result is that the text plays longer than the number of pages suggests.

The Appendix should at least be skimmed before working through the main chapters; it consists of a list of notation and terminology that is, for the most part, *not* defined in the main chapters.

A list of references for background and supplementary reading is provided.

Finally, there is a set of indexes that may aid in navigating the book: the first is an index of examples; the second is an index of exercises; the third is an index of problems; the fourth is an index of results (i.e., lemmas, theorems, propositions, corollaries); the last is an index of algorithms.

We begin with an Introduction to the *mind set* of combinatorial optimization and the polyhedral viewpoint.

Chapter 0 contains "prerequisite" results concerning polytopes and linear programming. Although the material of Chapter 0 is prerequisite, most linear-programming courses will not have covered all of this chapter. When I have taught from this book, I start right in with Chapter 1 after working through the Introduction. Then, as needed while working through Chapters 1–8, I ask students to read, or I cover in class, parts of Chapter 0. In particular, Section 0.5 is needed for Sections 1.7, 3.4, and 4.2; Section 0.2 is needed for Sections 1.7, 4.3, 4.4, and 5.2; Section 0.6 is needed for Section 7.3; and Sections 0.3 and 0.7 are needed for Section 6.3.

Although Chapter 0 does not contain a comprehensive treatment of linear programming, by adding some supplementary material on (1) practical implementation details for the simplex method, (2) the ellipsoid method, and (3) interior-point methods, this chapter can be used as the core of a more full treatment of linear programming.

The primary material starts with Chapter 1. In this chapter, we concentrate on matroids and the greedy algorithm. Many of the central ideas that come up later, like submodularity and methods of polyhedral combinatorics, are first explored in this chapter.

Chapter 2, in which we develop the basic algorithms to calculate minimum-weight dipaths, is somewhat of a digression. However, minimum-weight dipaths and the associated algorithms are important building blocks for other algorithms.

In Chapter 3, we discuss the problem of finding maximum-cardinality, and, more generally, maximum-weight sets that are independent in two matroids on a common ground set. The algorithms and polyhedral results are striking in their beauty and complexity.

The subject of Chapter 4 is matchings in graphs. As in the previous chapter, striking algorithms and polyhedral results are presented. We discuss some applications of matching to other combinatorial-optimization problems.

The subjects of Chapters 3 and 4 can be viewed as two different generalizations of the problem of finding maximum-cardinality and maximum-weight matchings in *bipartite* graphs. We find that König's min/max theorem, as well as the algorithmic and polyhedral results, generalize in quite different ways.

In Chapter 5, we discuss the maximum-flow problem for digraphs and related cut problems. Although the topic seems less intricate than those of the two previous chapters, we discuss the seminal method of Edmonds and Karp that is used to produce an efficient algorithm. Also, the methods of this chapter relate to those of Chapter 2.

In Chapter 6, we study cutting-plane methods for integer programming. We begin with the fundamental idea of taking nonnegative linear combinations and rounding. The details of Gomory's finite cutting-plane algorithm are described. There is a general discussion of methods for tightening integer-programming formulations. Examples of special-purpose cutting-plane methods for combinatorial-optimization problems are also given.

In Chapter 7, Branch-&-Bound methods for solving discrete-optimization problems are described. The general framework is not very interesting from a mathematical point of view, but the bounding methods, for example, can be quite sophisticated. Also, Branch-&-Bound is a very useful practical technique for solving combinatorial-optimization problems.

In Chapter 8, we discuss optimization of submodular functions. Many of the problems that were treated in the earlier chapters can be viewed as problems of minimizing or maximizing particular submodular functions. Although the efficient algorithms for minimizing general submodular functions are not described, it is valuable to explore the unifying role of submodular functions.

And there it ends. A sequel to this book would study (1) semidefinite programming formulations of combinatorial-optimization problems and associated interior-point algorithms for the solution of the relaxations, (2) efficient approximation algorithms with performance guarantees for combinatorial-optimization problems, (3) algebraic methods for integer programming, (4) and much more on submodular optimization. The practical significance of these subjects has yet to be firmly established, but the theory is great!

I thank those who first taught me about combinatorics and optimization at Cornell: Lou Billera, Bob Bland, Jack Edmonds, George Nemhauser, Mike Todd, and Les Trotter. Further thanks are due to Carl Lee, François Margot, and students at the University of Kentucky and New York University who worked through drafts of this book; they made many valuable suggestions, most of which I stubbornly ignored.

Finally, this project would never have been completed without the firm yet compassionate guidance of Lauren Cowles, Caitlin Doggart, Katie Hew, and Lara Zoble of Cambridge University Press and Michie Shaw of TechBooks.

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