Geometric differentiation

Geometric differentiation

for the *intelligence* of curves and surfaces

Second edition

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University Printing House, Cambridge CB2 8BS, United Kingdom

Cambridge University Press is part of the University of Cambridge.

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www.cambridge.org
Information on this title: www.cambridge.org/9780521810401

First published 1994
Second edition 2001

A catalogue record for this publication is available from the British Library

ISBN 978-0-521-81040-1 Hardback
ISBN 978-0-521-00264-6 Paperback

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Cambridge University Press
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Introduction

This book is concerned with the local differential geometry of smooth curves and surfaces in Euclidean space. This is a topic which is generally compressed into a short introductory chapter in any standard work on differential geometry, but nevertheless has a surprising richness, with new areas still being explored.

The initial impetus to look afresh at this subject was René Thom's choice ((1975), but circulating in draft for several years previous to this data) of the term 'umbilic', borrowed from differential geometry, to describe certain of the elementary catastrophes, and the realisation that Darboux's classification (1896) of the umbilics of a surface, itself not sufficiently known, was only part of the story (Porteous, 1971). In particular the *ridges* of a surface are important features that are almost entirely disregarded in the standard texts, though certain of them are familiar to structural geologists as the *hinge lines* of folds in strata (Ramsay, 1967). One of the few places where they get serious mention in the literature is in the work (1904) of A. Gullstrand, who was awarded the Nobel Prize for Physiology and Medicine in 1911 for his work on the accommodation of the eye lens, in which work he had to create the necessary fourth order differential geometry to explain the relevant optics. His Prize Lecture (1911) makes amusing reading. Recently they have been rediscovered by workers on face recognition and the interpretation of magnetic resonance scans of the surface of the brain.

The work of singularity theorists, starting with H. Whitney, R. Thom and J. Mather and extended by many others, notably the Russian School under the leadership of V. I. Arnol'd, has richly developed the geometry of Taylor series of smooth functions, making accessible and familiar the classification of the higher order critical points of families of smooth functions. What we do here is to provide a treatment of surface theory which is in the spirit of singularity theory, yet stops short of the hard and powerful theorems of that subject. For this reason some of the things that we do are heuristic only, though with reference to where a fuller treatment is to be found.

A unifying theme throughout the book is the use of the family of distance-squared functions that relate a smooth submanifold of Euclidean space to the ambient space. The description of the critical points of these functions in terms of invariant expressions involving the higher derivatives of these functions provides a high road into the interesting geometry.

The first chapter plunges straight into the interrelationship of a smooth parametric plane curve with its *evolute* or *focal curve*, a topic first studied in detail by Huygens some years before the advent of the differential calculus made this a first exercise in that subject. It is one that is still important in the practical theory of the *offsets* or *parallels* of a plane curve. As Huygens realised, these are recoverable from the evolute by unwinding a string from it or, equivalently, by rolling a ruler along it, a pregnant idea that bears fruit later in the theory of space curves and surfaces and their evolutes. Several characterisations are given of the *curvature* and the *vertices* of a curve. The latter topic is the first place where third derivatives make their entry. We learn how to locate the vertices without having to differentiate the explicit but somewhat unattractive formula for the curvature. As a digression from the main theme we apply these ideas briefly to some basic concepts of *plane kinematics* that have application in the theory of mechanisms, namely the *inflection circle*, the *Ball point*, the *cubic of stationary curvature* and the *Burmester points* of the instantaneous motion of a plane moving rigidly over the plane. A related subject, though perhaps not obviously so, is the description of the *caustics* of the light from a point source reflected from a curved mirror, a subject richly reexamined recently by Bruce, Giblin and Gibson (1981). The analogous theory of spherical kinematics is briefly sketched in a later chapter, prefaced by a short chapter in which some basic facts of elementary geometry are recalled. Spherical kinematics has application to the description of the motion of plates over the surface of the Earth in modern plate tectonics.

Next comes the theory of *space curves*, introducing the *space evolute* of a space curve and the *tangent developable* of the space evolute, the *focal surface* of the original curve. This is presented before the derivation of the classical *Serret–Frenet equations* for a space curve, usually given pride of place. The latter require the theoretically convenient but practically inconvenient use of a unit-speed parametrisation for the curve. Also defined are the *parallels* to a space curve, the involutes of its evolute, and the *focal curves* of a space curve, shown to be geodesics on its focal surface.

Surface theory, depending as it does on the differential calculus of functions of several variables, requires familiarity not only with *quadratic* but also with *cubic forms* in two variables. A chapter interpolated here provides the necessary facts. Two further short chapters follow, one on probe analysis as a diagnostic technique for handling singularities of maps and the other on contact. The way is then open to describe the *focal surface* of a regular smooth parametric surface in three-dimensional Euclidean space.

Topics discussed in detail include the *first and second fundamental forms*, the *principal centres of curvature* and *principal curvatures* of the surface and the induced grid on the surface formed by the two mutually orthogonal families of *lines of curvature*. These only involve the first and second derivatives of the parametrisation. It is in the description of the *ridges* of the surface and the associated *ribs* or cuspidal edges of the focal surface and the rich geometry around the *umbilics* of the surface, those points where the principal curvatures agree and the surface is most nearly spherical, that the third and higher derivatives become important and the insights of singularity theory prove of greatest value.

Other topics discussed include various aspects of the *Gauss map* of a surface, including the *parabolic line*, the *cusps of Gauss* and Gauss' 'excellent' theorem, in Greek–Latin the *theorema egregium*.

The study of the focal surface of a surface is rounded off by a detailed examination of the inverse construction of the involutes or evolvents of a geodesic foliation on a regular surface in three-dimensional Euclidean space with emphasis on what happens when the curves of the foliation possess linear points, a subject full of surprises that has had close attention from the Russian school of singularity theorists under the leadership of V.I. Arnol'd, and where the full story has only very recently become clear.

Our treatment, based on the study of the family of distance-squared functions, has emphasised the role of spheres in surface theory. The circles associated to a surface are also of great interest. Here again the initial steps recovering the focal surface are classical, but much of the higher order material seems to be new, some of it to be found in the thesis of James Montaldi (1986a). Remarks on the bumpy circular contours round an umbilical hill-top led to the delightful work of Stelios Markatis (1980) on bumpy spheres, outlined here in a final chapter that considers some other important examples also.

The title of the book has been chosen to indicate that its purpose is to make geometrical the basic properties of the derivatives of differential maps. It is written to be accessible to final-year honours students or to first-year postgraduates, keen to add some geometry to the standard linear algebra and several variables calculus that in one form or another they will already have met. A non-standard feature which browsers must take note of is a notational one. For practical reasons that will become evident when working with third and fourth derivatives the traditional d to denote differentiation is only used occasionally. Instead differentiation is denoted by *subscripts*. Thus for a smooth map $f : X \rightarrowtail Y$ between real vector spaces X and Y (where the tail on the arrow indicates that the domain is an open subset of the source vector space) the first derivative is the map $f_1 : X \rightarrowtail L(X, Y)$, that associates to each $a \in \operatorname{dom} f$ the linear map $f_1(a) : X \rightarrow Y$ that (up to a constant) best approximates f at a, the second derivative is $f_2 : X \rightarrowtail L(X,(X, Y))$, associating to each $a \in X$ the *twice* linear (rather than *bilinear*) map $f_2(a) = (f_1)_1(a)$, and so on.

Much of the material has been class tested from time to time over the last fifteen years. It has been the basis of a number of expository lectures, for example to the Archimedeans in Cambridge on several occasions, to colleagues in singularity theory at the AMS symposium at Arcata in 1981 (Porteous 1983a,b), to specialists in computer vision at a Rank Prize Fund symposium at Liverpool in 1986 and to the Oxford group in 1988, and to specialists in computer-aided design at symposia on the Theory of Surfaces organised by the Institute of Mathematics and its Applications at Cardiff in 1986 and Oxford in 1988 (Porteous, 1987a, 1987b, 1989).

Much of the work has been done in collaboration with students at Liverpool, notably Stelios Markatis (1980), James Montaldi (1983), Alex Flegmann (1985) and Richard Morris (1990). My thanks are also due to numerous colleagues and friends for their interest and comment, especially Bill Bruce, Peter Giblin, Chris Gibson, Peter Newstead and Terry Wall and students Helen Chappell and Neil Kirk.

Especial thanks are due to Peter Ackerley, who has crafted many of the figures, and to Peter Giblin and Richard Morris for those produced on the computer. I am also most grateful to John Robinson of Yeovil for

permitting me to make use of his sculpture *Eternity* to grace this book. For his remarkable story and pictures of all his abstract sculptures see Robinson (1992).

Я благодарен также В. И. Арнольду, чья помошь была мие очень полезна.

Liverpool, 1993 Ian R. Porteous

Almost as soon as the first edition of this book appeared in print there were two developments which I am delighted to be able to incorporate into this new edition, which preserves as far as possible the pagination of the original, but provides two chapters of additional material.

In the first place Richard Morris, Mike Puddephat and Tom Barrick have helped to resolve what happens to the subparabolic lines of a surface, or *flexcords* as I now prefer to call them, at generic, and near-generic births of umbilics, there being related work also by Bill Bruce, Peter Giblin and Farid Tari, using the machinery of singularity theory. This corrects and amplifies statements made on page 228 of the first edition and provides the material of a new Chapter 17.

Secondly, a remarkable 68-page paper by V.I. Arnol'd appeared in 1995 on the geometry of curves on S^2, which complements and extends the material of Chapter 5. Details are to be found in a new Chapter 18, which also contains new results on curves and surfaces in S^3.

Initial drafts of both these chapters have appeared as Porteous and Puddephat (2000) and Porteous (1999), respectively. I am grateful both to Springer and to the Banach Institute for permitting me to reuse this material, substantially unaltered.

Interest in the ridges of a surface, in particular, has continued among those working in medical imaging and in CAD. What often is required is the ability to *landmark* a surface that may be slowly changing in time. Lines of curvature and geodesics are of no direct assistance, as they reform rather than deform as the surface changes. Besides the umbilics and the parabolic line, familiar to all differential geometers, it is the lesser-known ridges and hardly known flexcords, and special points on them, that fill the need.

The opportunity has of course been taken to correct a number of misprints in the first printing, many of which have been brought to my attention by diligent readers, to all of whom I am most grateful.

Liverpool, 2000 Ian R. Porteous