An Introduction to Numerical Analysis

Endre Süli and David F. Mayers

University of Oxford

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Preface

This book has grown out of printed notes which accompanied lectures given by ourselves and our colleagues over many years to undergraduate mathematicians at Oxford. During those years the contents and the arrangement of the lectures have changed substantially, and this book has a wider scope than is currently taught. It contains mathematics which, in an ideal world, would be part of the equipment of any well-educated mathematician.

Numerical analysis is the branch of mathematics concerned with the theoretical foundations of numerical algorithms for the solution of problems arising in scientific applications. The subject addresses a variety of questions ranging from the approximation of functions and integrals to the approximate solution of algebraic, transcendental, differential and integral equations, with particular emphasis on the stability, accuracy, efficiency and reliability of numerical algorithms. The purpose of this book is to provide an elementary introduction into this active and exciting field, and is aimed at students in the second year of a university mathematics course.

The book addresses a wide range of numerical problems in algebra and analysis. Chapter 2 deals with the solution of systems of linear equations, a process which can be completed in a finite number of arithmetical operations. In the rest of the book the solution of a problem is sought as the limit of an infinite sequence; in that sense the output of the numerical algorithm is an 'approximate' solution. This need not, however, mean any relaxation of the usual standards of rigorous analysis. The idea of convergence of a sequence of real numbers (x_n) to a real number ξ is very familiar: given any positive value of ε there exists a positive integer N_0 such that $|x_n - \xi| < \varepsilon$ for all n such that $n > N_0$. In such a situation one can obtain as accurate an approximation to ξ as required by calculating sufficiently many members of the sequence, or just one member, sufficiently far along. A 'pure mathematician' would prefer the exact answer, ξ, but the sorts of guaranteed accurate approximations which will be discussed here are entirely satisfactory in real-life applications.

Numerical analysis brings two new ideas to the usual discussion of convergence of sequences. First, we need, not just the existence of N_0, but a good estimate of how large it is; and it may be too large for practical calculations. Second, rather than being asked for the limit of a given sequence, we are usually given the existence of the limit ξ (or its approximate location on the real line) and then have to construct a sequence which converges to it. If the rate of convergence is slow, so that the value of N_0 is large, we must then try to construct a better sequence, one that converges to ξ more rapidly. These ideas have direct applications in the solution of a single nonlinear equation in Chapter 1, the solution of systems of nonlinear equations in Chapter 4 and the calculation of the eigenvalues and eigenvectors of a matrix in Chapter 5.

The next six chapters are concerned with polynomial approximation, and show how, in various ways, we can construct a polynomial which approximates, as accurately as required, a given continuous function. These ideas have an obvious application in the evaluation of integrals, where we calculate the integral of the approximating polynomial instead of the integral of the given function.

Finally, Chapters 12 to 14 deal with the numerical solution of ordinary differential equations, with Chapter 14 presenting the fundamentals of the finite element method. The results of Chapter 14 can be readily extended to linear second-order partial differential equations.

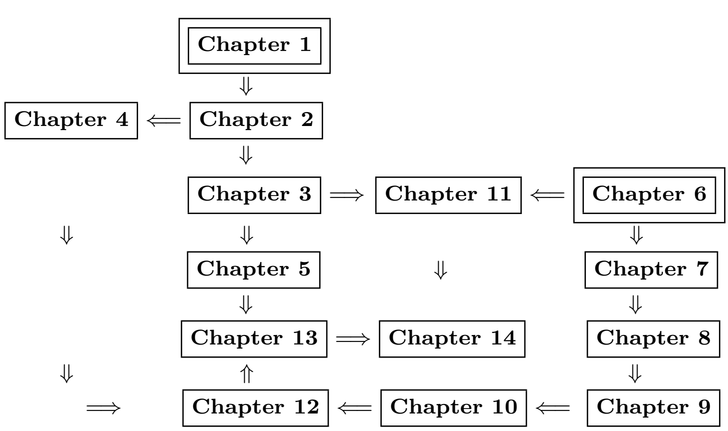
We have tried to make the coverage as complete as is consistent with remaining quite elementary. The limitations of size are most obvious in Chapter 12 on the solution of initial value problems for ordinary differential equations. This is an area where a number of excellent books are available, at least one of which is published in two weighty volumes. Chapter 12 does not describe or analyse anything approaching all the available methods, but we hope we have included some of those in most common use.

There is a selection of Exercises at the end of each chapter. All these exercises are theoretical; students are urged to apply all the methods described to some simple examples to see what happens. A few of the exercises will be found to require some heavy algebraic manipulation; these have been included because we assume that readers will have access to some computer algebra system such as Maple or Mathematica, which then make the algebraic work almost trivial. Those involved in teaching courses based on this book may obtain copies of LaTeX files containing solutions to these exercises by applying to the publisher by email (`solutions@cambridge.org`). Although the material presented in this book does not presuppose the reader's acquaintance with mathematical software packages, the importance of these cannot be overemphasised. In Appendix B, a brief set of pointers is provided to relevant software repositories.

Our treatment is intended to maintain a reasonably high standard of rigour, with many theorems and formal proofs. The main prerequisite is therefore some familiarity with elementary real analysis. Appendix A lists the standard theorems (labelled **Theorem A.1**, **A.2**, ..., **A7**) which are used in the book, together with proofs of one or two of them which might be less familiar. Some knowledge of basic matrix algebra is assumed. We have also used some elementary ideas from the theory of normed linear spaces in a number of places; complete definitions and examples are given. Some prior knowledge of these areas would be helpful, although not essential.

The chart below indicates how the chapters of the book are interrelated. They show, in particular, how Chapters 1 to 5 form a largely self-contained unit, as do Chapters 6 to 10.

Roadmap of the book



We have included some historical notes throughout the book. As well as hoping to stimulate an interest in the development of the subject, these notes show how wide a historical range even this elementary book covers. Many of the methods were developed by the great mathematicians of the seventeenth and eighteenth centuries, including Newton, Euler and Gauss, but what is usually known as Gaussian elimination for the solution of systems of linear equations was known to the Chinese two thousand years ago. At the other end of the historical scale, the analysis of the eigenvalue problem, and the numerical solution of differential equations, are much more recent, and are due to mathematicians who are still very much alive. Many of our historical notes are based on the excellent biographical database at the history of mathematics website

`http://www-history.mcs.st-andrews.ac.uk/history/`

We have tried to eradicate as many typographical errors from the text as possible; however, we are mindful that some may have escaped our attention. We plan to post any typos reported to us on

`http://web.comlab.ox.ac.uk/oucl/work/endre.suli/index.html`

We wish to express our gratitude to Professor Bill Morton for setting us off on this *tour de force*, to David Tranah at Cambridge University Press for encouraging us to persist with the project, and to the staff of the Press for not only improving the appearance of the book and eliminating a number of typographical errors, but also for correcting and improving some of our mathematics. We also wish to thank our colleagues at the Oxford University Computing Laboratory, particularly Nick Trefethen, Mike Giles and Andy Wathen, for keeping our spirits up, and to Paul Houston at the Department of Mathematics and Computer Science of the University of Leicester for his help with the final example in the book.

Above all, we are grateful to our families for their patience, support and understanding: this book is dedicated to them.

ES & DFM *Oxford, September 2002.*