

REAL ANALYSIS AND PROBABILITY

This much admired textbook, now reissued in paperback, offers a clear exposition of modern probability theory and of the interplay between the properties of metric spaces and probability measures.

The first half of the book gives an exposition of real analysis: basic set theory, general topology, measure theory, integration, an introduction to functional analysis in Banach and Hilbert spaces, convex sets and functions, and measure on topological spaces. The second half introduces probability based on measure theory, including laws of large numbers, ergodic theorems, the central limit theorem, conditional expectations, and martingale convergence. A chapter on stochastic processes introduces Brownian motion and the Brownian bridge.

The new edition has been made even more self-contained than before; it now includes early in the book a foundation of the real number system and the Stone-Weierstrass theorem on uniform approximation in algebras of functions. Several other sections have been revised and improved, and the extensive historical notes have been further amplified. A number of new exercises, and hints for solution of old and new ones, have been added.

R. M. Dudley is Professor of Mathematics at the Massachusetts Institute of Technology in Cambridge, Massachusetts.

CAMBRIDGE STUDIES IN ADVANCED MATHEMATICS

Editorial Board:

B. Bollobas, W. Fulton, A. Katok, F. Kirwan, P. Sarnak

Already published

- 17 W. Dicks & M. Dunwoody *Groups acting on graphs*
- 18 L.J. Corwin & F.P. Greenleaf *Representations of nilpotent Lie groups and their applications*
- 19 R. Fritsch & R. Piccinini *Cellular structures in topology*
- 20 H. Klingen *Introductory lectures on Siegel modular forms*
- 21 P. Koosis *The logarithmic integral II*
- 22 M.J. Collins *Representations and characters of finite groups*
- 24 H. Kunita *Stochastic flows and stochastic differential equations*
- 25 P. Wojtaszczyk *Banach spaces for analysts*
- 26 J.E. Gilbert & M.A.M. Murray *Clifford algebras and Dirac operators in harmonic analysis*
- 27 A. Frohlich & M.J. Taylor *Algebraic number theory*
- 28 K. Goebel & W.A. Kirk *Topics in metric fixed point theory*
- 29 J.F. Humphreys *Reflection groups and Coxeter groups*
- 30 D.J. Benson *Representations and cohomology I*
- 31 D.J. Benson *Representations and cohomology II*
- 32 C. Allday & V. Puppe *Cohomological methods in transformation groups*
- 33 C. Soule et al. *Lectures on Arakelov geometry*
- 34 A. Ambrosetti & G. Prodi *A primer of nonlinear analysis*
- 35 J. Palis & F. Takens *Hyperbolicity, stability and chaos at homoclinic bifurcations*
- 37 Y. Meyer *Wavelets and operators I*
- 38 C. Weibel *An introduction to homological algebra*
- 39 W. Bruns & J. Herzog *Cohen-Macaulay rings*
- 40 V. Snaith *Explicit Brauer induction*
- 41 G. Laumon *Cohomology of Drinfeld modular varieties I*
- 42 E.B. Davies *Spectral theory and differential operators*
- 43 J. Diestel, H. Jarchow, & A. Tonge *Absolutely summing operators*
- 44 P. Mattila *Geometry of sets and measures in Euclidean spaces*
- 45 R. Pinsky *Positive harmonic functions and diffusion*
- 46 G. Tenenbaum *Introduction to analytic and probabilistic number theory*
- 47 C. Peskine *An algebraic introduction to complex projective geometry*
- 48 Y. Meyer & R. Coifman *Wavelets*
- 49 R. Stanley *Enumerative combinatorics I*
- 50 I. Porteous *Clifford algebras and the classical groups*
- 51 M. Audin *Spinning tops*
- 52 V. Jurdjevic *Geometric control theory*
- 53 H. Volklein *Groups as Galois groups*
- 54 J. Le Potier *Lectures on vector bundles*
- 55 D. Bump *Automorphic forms and representations*
- 56 G. Laumon *Cohomology of Drinfeld modular varieties II*
- 57 D.M. Clark & B.A. Davey *Natural dualities for the working algebraist*
- 58 J. McCleary *A user's guide to spectral sequences II*
- 59 P. Taylor *Practical foundations of mathematics*
- 60 M.P. Brodmann & R.Y. Sharp *Local cohomology*
- 61 J.D. Dixon et al. *Analytic pro-P groups*
- 62 R. Stanley *Enumerative combinatorics II*
- 63 R.M. Dudley *Uniform central limit theorems*
- 64 J. Jost & X. Li-Jost *Calculus of variations*
- 65 A.J. Berrick & M.E. Keating *An introduction to rings and modules*
- 66 S. Morosawa *Holomorphic dynamics*
- 67 A.J. Berrick & M.E. Keating *Categories and modules with K-theory in view*
- 68 K. Sato *Levy processes and infinitely divisible distributions*
- 69 H. Hida *Modular forms and Galois cohomology*
- 70 R. Iorio & V. Iorio *Fourier analysis and partial differential equations*
- 71 R. Blei *Analysis in integer and fractional dimensions*
- 72 F. Borceaux & G. Janelidze *Galois theories*
- 73 B. Bollobas *Random graphs*

Cambridge University Press
052180972X - Real Analysis and Probability
R. M. Dudley
Frontmatter
[More information](#)

REAL ANALYSIS AND PROBABILITY

R. M. DUDLEY

Massachusetts Institute of Technology



CAMBRIDGE
UNIVERSITY PRESS

Cambridge University Press
052180972X - Real Analysis and Probability
R. M. Dudley
Frontmatter
[More information](#)

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK
40 West 20th Street, New York, NY 10011-4211, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa
<http://www.cambridge.org>

© R. M. Dudley 2002

This book is in copyright. Subject to statutory exception
and to the provisions of relevant collective licensing agreements,
no reproduction of any part may take place without
the written permission of Cambridge University Press.

First published 1989 by Wadsworth, Inc.
Cambridge University Press edition published 2002

Printed in the United Kingdom at the University Press, Cambridge

Typeface Times Roman 10.25/13 pt. *System* L^AT_EX 2_ε [TB]

A catalog record for this book is available from the British Library.

Library of Congress Cataloging in Publication Data available

ISBN 0 521 80972 X hardback
ISBN 0 521 00754 2 paperback

Contents

<i>Preface to the Cambridge Edition</i>	<i>page ix</i>
1 Foundations; Set Theory	1
1.1 Definitions for Set Theory and the Real Number System	1
1.2 Relations and Orderings	9
*1.3 Transfinite Induction and Recursion	12
1.4 Cardinality	16
1.5 The Axiom of Choice and Its Equivalents	18
2 General Topology	24
2.1 Topologies, Metrics, and Continuity	24
2.2 Compactness and Product Topologies	34
2.3 Complete and Compact Metric Spaces	44
2.4 Some Metrics for Function Spaces	48
2.5 Completion and Completeness of Metric Spaces	58
*2.6 Extension of Continuous Functions	63
*2.7 Uniformities and Uniform Spaces	67
*2.8 Compactification	71
3 Measures	85
3.1 Introduction to Measures	85
3.2 Semirings and Rings	94
3.3 Completion of Measures	101
3.4 Lebesgue Measure and Nonmeasurable Sets	105
*3.5 Atomic and Nonatomic Measures	109
4 Integration	114
4.1 Simple Functions	114
*4.2 Measurability	123
4.3 Convergence Theorems for Integrals	130

4.4	Product Measures	134
*4.5	Daniell-Stone Integrals	142
5	L^p Spaces; Introduction to Functional Analysis	152
5.1	Inequalities for Integrals	152
5.2	Norms and Completeness of L^p	158
5.3	Hilbert Spaces	160
5.4	Orthonormal Sets and Bases	165
5.5	Linear Forms on Hilbert Spaces, Inclusions of L^p Spaces, and Relations Between Two Measures	173
5.6	Signed Measures	178
6	Convex Sets and Duality of Normed Spaces	188
6.1	Lipschitz, Continuous, and Bounded Functionals	188
6.2	Convex Sets and Their Separation	195
6.3	Convex Functions	203
*6.4	Duality of L^p Spaces	208
6.5	Uniform Boundedness and Closed Graphs	211
*6.6	The Brunn-Minkowski Inequality	215
7	Measure, Topology, and Differentiation	222
7.1	Baire and Borel σ -Algebras and Regularity of Measures	222
*7.2	Lebesgue's Differentiation Theorems	228
*7.3	The Regularity Extension	235
*7.4	The Dual of $C(K)$ and Fourier Series	239
*7.5	Almost Uniform Convergence and Lusin's Theorem	243
8	Introduction to Probability Theory	250
8.1	Basic Definitions	251
8.2	Infinite Products of Probability Spaces	255
8.3	Laws of Large Numbers	260
*8.4	Ergodic Theorems	267
9	Convergence of Laws and Central Limit Theorems	282
9.1	Distribution Functions and Densities	282
9.2	Convergence of Random Variables	287
9.3	Convergence of Laws	291
9.4	Characteristic Functions	298
9.5	Uniqueness of Characteristic Functions and a Central Limit Theorem	303
9.6	Triangular Arrays and Lindeberg's Theorem	315
9.7	Sums of Independent Real Random Variables	320

<i>Contents</i>	vii
*9.8 The Lévy Continuity Theorem; Infinitely Divisible and Stable Laws	325
10 Conditional Expectations and Martingales	336
10.1 Conditional Expectations	336
10.2 Regular Conditional Probabilities and Jensen's Inequality	341
10.3 Martingales	353
10.4 Optional Stopping and Uniform Integrability	358
10.5 Convergence of Martingales and Submartingales	364
*10.6 Reversed Martingales and Submartingales	370
*10.7 Subadditive and Superadditive Ergodic Theorems	374
11 Convergence of Laws on Separable Metric Spaces	385
11.1 Laws and Their Convergence	385
11.2 Lipschitz Functions	390
11.3 Metrics for Convergence of Laws	393
11.4 Convergence of Empirical Measures	399
11.5 Tightness and Uniform Tightness	402
*11.6 Strassen's Theorem: Nearby Variables with Nearby Laws	406
*11.7 A Uniformity for Laws and Almost Surely Converging Realizations of Converging Laws	413
*11.8 Kantorovich-Rubinstein Theorems	420
*11.9 U -Statistics	426
12 Stochastic Processes	439
12.1 Existence of Processes and Brownian Motion	439
12.2 The Strong Markov Property of Brownian Motion	450
12.3 Reflection Principles, The Brownian Bridge, and Laws of Suprema	459
12.4 Laws of Brownian Motion at Markov Times: Skorohod Imbedding	469
12.5 Laws of the Iterated Logarithm	476
13 Measurability: Borel Isomorphism and Analytic Sets	487
*13.1 Borel Isomorphism	487
*13.2 Analytic Sets	493
Appendix A Axiomatic Set Theory	503
A.1 Mathematical Logic	503
A.2 Axioms for Set Theory	505

A.3 Ordinals and Cardinals	510
A.4 From Sets to Numbers	515
Appendix B Complex Numbers, Vector Spaces, and Taylor's Theorem with Remainder	521
Appendix C The Problem of Measure	526
Appendix D Rearranging Sums of Nonnegative Terms	528
Appendix E Pathologies of Compact Nonmetric Spaces	530
<i>Author Index</i>	541
<i>Subject Index</i>	546
<i>Notation Index</i>	554

Preface to the Cambridge Edition

This is a text at the beginning graduate level. Some study of intermediate analysis in Euclidean spaces will provide helpful background, but in this edition such background is not a formal prerequisite. Efforts to make the book more self-contained include inserting material on the real number system into Chapter 1, adding a treatment of the Stone-Weierstrass theorem, and generally eliminating references for proofs to other books except at very few points, such as some complex variable theory in Appendix B.

Chapters 1 through 5 provide a one-semester course in real analysis. Following that, a one-semester course on probability can be based on Chapters 8 through 10 and parts of 11 and 12. Starred paragraphs and sections, such as those found in Chapter 6 and most of Chapter 7, are called on rarely, if at all, later in the book. They can be skipped, at least on first reading, or until needed.

Relatively few proofs of less vital facts have been left to the reader. I would be very glad to know of any substantial unintentional gaps or errors. Although I have worked and checked all the problems and hints, experience suggests that mistakes in problems, and hints that may mislead, are less obvious than errors in the text. So take hints with a grain of salt and perhaps make a first try at the problems without using the hints.

I looked for the best and shortest available proofs for the theorems. Short proofs that have appeared in journal articles, but in few if any other textbooks, are given for the completion of metric spaces, the strong law of large numbers, the ergodic theorem, the martingale convergence theorem, the subadditive ergodic theorem, and the Hartman-Wintner law of the iterated logarithm.

Around 1950, when Halmos' classic *Measure Theory* appeared, the more advanced parts of the subject headed toward measures on locally compact spaces, as in, for example, §7.3 of this book. Since then, much of the research in probability theory has moved more in the direction of metric spaces. Chapter 11 gives some facts connecting metrics and probabilities which follow the newer trend. Appendix E indicates what can go wrong with measures

on (locally) compact nonmetric spaces. These parts of the book may well not be reached in a typical one-year course but provide some distinctive material for present and future researchers.

Problems appear at the end of each section, generally increasing in difficulty as they go along. I have supplied hints to the solution of many of the problems. There are a lot of new or, I hope, improved hints in this edition.

I have also tried to trace back the history of the theorems to give credit where it is due. Historical notes and references, sometimes rather extensive, are given at the end of each chapter. Many of the notes have been augmented in this edition and some have been corrected. I don't claim, however, to give the last word on any part of the history.

The book evolved from courses given at M.I.T. since 1967 and in Aarhus, Denmark, in 1976. For valuable comments I am glad to thank Ken Alexander, Deborah Allinger, Laura Clemens, Ken Davidson, Don Davis, Persi Diaconis, Arnout Eikeboom, Sy Friedman, David Gillman, José Gonzalez, E. Griffor, Leonid Grinblat, Dominique Haughton, J. Hoffmann-Jørgensen, Arthur Mattuck, Jim Munkres, R. Proctor, Nick Reingold, Rae Shortt, Dorothy Maharam Stone, Evangelos Tabakis, Jin-Gen Yang, and other students and colleagues.

For helpful comments on the first edition I am thankful to Ken Brown, Justin Corvino, Charles Goldie, Charles Hadlock, Michael Jansson, Suman Majumdar, Rimas Norvaiša, Mark Pinsky, Andrew Rosalsky, the late Rae Shortt, and Dewey Tucker. I especially thank Andries Lenstra and Valentin Petrov for longer lists of suggestions. Major revisions have been made to §10.2 (regular conditional probabilities) and in Chapter 12 with regard to Markov times.

R. M. Dudley