REAL ANALYSIS AND PROBABILITY

This much admired textbook, now reissued in paperback, offers a clear exposition of modern probability theory and of the interplay between the properties of metric spaces and probability measures.

The first half of the book gives an exposition of real analysis: basic set theory, general topology, measure theory, integration, an introduction to functional analysis in Banach and Hilbert spaces, convex sets and functions, and measure on topological spaces. The second half introduces probability based on measure theory, including laws of large numbers, ergodic theorems, the central limit theorem, conditional expectations, and martingale convergence. A chapter on stochastic processes introduces Brownian motion and the Brownian bridge.

The new edition has been made even more self-contained than before; it now includes early in the book a foundation of the real number system and the Stone-Weierstrass theorem on uniform approximation in algebras of functions. Several other sections have been revised and improved, and the extensive historical notes have been further amplified. A number of new exercises, and hints for solution of old and new ones, have been added.

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Contents

Preface to the Cambridge Edition

This is a text at the beginning graduate level. Some study of intermediate analysis in Euclidean spaces will provide helpful background, but in this edition such background is not a formal prerequisite. Efforts to make the book more self-contained include inserting material on the real number system into Chapter 1, adding a treatment of the Stone-Weierstrass theorem, and generally eliminating references for proofs to other books except at very few points, such as some complex variable theory in Appendix B.

Chapters 1 through 5 provide a one-semester course in real analysis. Following that, a one-semester course on probability can be based on Chapters 8 through 10 and parts of 11 and 12. Starred paragraphs and sections, such as those found in Chapter 6 and most of Chapter 7, are called on rarely, if at all, later in the book. They can be skipped, at least on first reading, or until needed.

Relatively few proofs of less vital facts have been left to the reader. I would be very glad to know of any substantial unintentional gaps or errors. Although I have worked and checked all the problems and hints, experience suggests that mistakes in problems, and hints that may mislead, are less obvious than errors in the text. So take hints with a grain of salt and perhaps make a first try at the problems without using the hints.

I looked for the best and shortest available proofs for the theorems. Short proofs that have appeared in journal articles, but in few if any other textbooks, are given for the completion of metric spaces, the strong law of large numbers, the ergodic theorem, the martingale convergence theorem, the subadditive ergodic theorem, and the Hartman-Wintner law of the iterated logarithm.

Around 1950, when Halmos' classic *Measure Theory* appeared, the more advanced parts of the subject headed toward measures on locally compact spaces, as in, for example, §7.3 of this book. Since then, much of the research in probability theory has moved more in the direction of metric spaces. Chapter 11 gives some facts connecting metrics and probabilities which follow the newer trend. Appendix E indicates what can go wrong with measures on (locally) compact nonmetric spaces. These parts of the book may well not be reached in a typical one-year course but provide some distinctive material for present and future researchers.

Problems appear at the end of each section, generally increasing in difficulty as they go along. I have supplied hints to the solution of many of the problems. There are a lot of new or, I hope, improved hints in this edition.

I have also tried to trace back the history of the theorems to give credit where it is due. Historical notes and references, sometimes rather extensive, are given at the end of each chapter. Many of the notes have been augmented in this edition and some have been corrected. I don't claim, however, to give the last word on any part of the history.

The book evolved from courses given at M.I.T. since 1967 and in Aarhus, Denmark, in 1976. For valuable comments I am glad to thank Ken Alexander, Deborah Allinger, Laura Clemens, Ken Davidson, Don Davis, Persi Diaconis, Arnout Eikeboom, Sy Friedman, David Gillman, José Gonzalez, E. Griffor, Leonid Grinblat, Dominique Haughton, J. Hoffmann-Jørgensen, Arthur Mattuck, Jim Munkres, R. Proctor, Nick Reingold, Rae Shortt, Dorothy Maharam Stone, Evangelos Tabakis, Jin-Gen Yang, and other students and colleagues.

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