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Random Graphs

Second Edition

BÉLA BOLLOBÁS
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and
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To Gabriella and Mark

'I learn so as to be contented.'

After the inscription on 'Tsukubai', the stone wash-basin in Ryoanji temple.

Contents

Preface to the Second Edition

The period since the publication of the first edition of this book has seen the theory of random graphs go from strength to strength. Indeed, its appearance happened to coincide with a watershed in the subject; the emergence in the subsequent few years of significant new ideas and tools, perhaps most notably concentration methods, has had a major impact. It could be argued that the subject is now qualitatively different, insofar as results that would previously have been inaccessible are now regarded as routine. Several long-standing issues have been resolved, including the value of the chromatic number of a random graph $G_{n,p}$, the existence of Hamilton cycles in random cubic graphs, and precise bounds on certain Ramsey numbers. It remains the case, though, that most of the material in the first edition of the book is vital for gaining an insight into the theory of random graphs.

It would be impossible, in a single volume, to prove all the substantial new results that we would wish to, so we have chosen instead to give brief descriptions and to sketch a number of proofs. In particular, we discuss the concentration of probability in various spaces, general phase transitions, the emergence of the giant component in a random graph process, the value and concentration of the chromatic number of a typical random graph, Hamilton cycles in random regular graphs and other sparse random graphs, the random assignment problem, the Ramsey number $R(3, t)$, jumbled and pseudorandom graphs, and models of small-world graphs. Six of the chapters have been thoroughly revised, namely those on the evolution of random graphs, long paths and cycles, the diameter, cliques and colouring, Ramsey theory, and explicit constructions.

It is hoped that this updated and expanded volume will provide a concise but thorough introduction to the theory of random graphs as it stands at the outset of the twenty-first century.

Memphis, Whitsunday 2001
Béla Bollobás

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Frontmatter
More information

Preface

The theory of random graphs was founded by Erdős and Rényi (1959, 1960, 1961*a*, *b*) after Erdős (1947, 1959, 1961) had discovered that probabilistic methods were often useful in tackling extremal problems in graph theory. Erdős proved, amongst other things, that for all natural numbers $g \geq 3$ and $k \geq 3$ there exist graphs with girth g and chromatic number k. Erdős did not construct such graphs explicitly but showed that most graphs in a certain class could be altered slightly to give the required examples.

This phenomenon was not entirely new in mathematics, although it was certainly surprising that probabilistic ideas proved to be so important in the study of such a simple finite structure as a graph. In analysis, Paley and Zygmund (1930*a*, *b*, 1932) had investigated random series of functions. One of their results was that if the real numbers c_n satisfy $\sum_{n=0}^{\infty} c_n^2 = \infty$ then $\sum_{n=0}^{\infty} \pm c_n \cos nx$ fails to be a Fourier–Lebesgue series for almost all choices of the signs. To exhibit a sequence of signs with this property is surprisingly difficult: indeed, no algorithm is known which constructs an appropriate sequence of signs from any sequence c_n with $\sum_{n=0}^{\infty} c_n^2 = \infty$. Following the initial work of Paley and Zygmund, random functions were investigated in great detail by Steinhaus (1930), Paley, Wiener and Zygmund (1932), Kac (1949), Hunt (1951), Ryll-Nardzewski (1953), Salem and Zygmund (1954), Dvoretzky and Erdős (1959) and many others. An excellent account of these investigations can be found in Kahane (1963, 1968). Probabilistic methods were also used by Littlewood and Offord (1938) to study the zeros of random polynomials and analytic functions. Some decades later, a simple but crucial combinatorial lemma from their work greatly influenced the study of random finite sets in vector spaces.

The first combinatorial structures to be studied probabilistically were tournaments, chiefly because random tournaments are intrinsically related to statistics. The study began with Kendall and Babington-Smith (1940) and a concise account of many of the results is given by Moon (1968*b*). Szele (1943) was, perhaps, the first to apply probabilistic ideas to extremal problems in combinatorics. He observed that some tournament of order n must have at least $n!/2^{n-1}$ Hamilton paths, because the expected number of Hamilton paths is $n!/2^{n-1}$. Once again, it is not easy to construct a tournament of order n with this many Hamilton paths. A little later, Erdős (1947) used a similar argument, based on the expected number of k-cliques in a graph of order n, to show that the Ramsey number $R(k)$ is greater than $2^{k/2}$.

Existence results based on probabilistic ideas can now be found in many branches of mathematics, especially in analysis, the geometry of Banach spaces, number theory, graph theory, combinatorics and computer science. Probabilistic methods have become an important part of the arsenal of a great many mathematicians. Nevertheless, this is only a beginning: in the next decade or two probabilistic methods are likely to become even more prominent. It is also likely that in the not too distant future it will be possible to carry out statistical analyses of more complicated systems. Mathematicians who are not interested in graphs for their own sake should view the theory of random graphs as a modest beginning from which we can learn a variety of techniques and can find out what kind of results we should try to prove about more complicated random structures.

As often happens in mathematics, the study of statistical aspects of graphs was begun independently and almost simultaneously by several authors, namely Ford and Uhlenbeck (1956), Gilbert (1957), Austin, Fagen, Penney and Riordan (1959) and Erdős and Rényi (1959). Occasionally all these authors are credited with the foundation of the theory of random graphs. However, this is to misconceive the nature of the subject. Only Erdős and Rényi introduced the methods which underlie the probabilistic treatment of random graphs. The other authors were all concerned with enumeration problems and their techniques were essentially deterministic.

There are two natural ways of estimating the proportion of graphs having a certain property. One may obtain exact formulae, using Pólya's enumeration theorem, generating functions, differential operators and the whole theory of combinatorial enumeration, and then either consider the task completed or else proceed to investigate the asymptotic behaviour of the exact but complicated formulae, which is often a daunting task.

This approach, whose spirit is entirely deterministic, was used in the first three papers mentioned above and has been carried further by numerous authors. Graphical enumeration is discussed in detail in the well known monograph of Harary and Palmer (1973) and the recent encyclopaedic treatise by Goulden and Jackson (1983). The connection between graphical enumeration and statistical mechanics is emphasized by Temperley (1981). The theory of enumeration is a beautiful, rich and rapidly developing area of combinatorics, but it has very little to do with the theory of random graphs.

The other approach was introduced by Erdős and Rényi and is expounded in this volume. It has only the slightest connection with enumeration. One is not interested in exact formulae but rather in approximating a variety of exact values by appropriate probability distributions and using probabilistic ideas, whenever possible. As shown by Erdős and Rényi, this probabilistic approach is often more powerful than the deterministic one.

It is often helpful to imagine a random graph as a living organism which evolves with time. It is born as a set of n isolated vertices and develops by successively acquiring edges at random. Our main aim is to determine at what stage of the evolution a particular property of the graph is likely to arise. To make this more precise, we shall consider the properties of a 'typical' graph in a probability space consisting of graphs of a particular type. The simplest such probability space consists of all graphs with a given set of n labelled vertices and M edges, and each such graph is assigned the same probability. Usually we shall write G_M for a random element of this probability space. Then, if H is any graph with the given vertex set and M edges, then $P(G_M = H) = 1/\binom{N}{M}$ where $N = \binom{n}{2}$.

In most cases we shall have a sequence of probability spaces. For each natural number n there will be a probability space consisting of graphs with exactly n vertices. We shall be interested in the properties of this space as $n \to \infty$. In this situation we shall say that a *typical element of our space has a property* Q when the probability that a random graph on n vertices has Q tends to 1 as $n \to \infty$. We also say that *almost every* (*a.e.*) graph has property Q. Thus almost every G_M has property Q if the proportion of graphs with this property tends to 1 as $n \to \infty$. Once we are given such probability spaces of graphs, numerous natural questions arise. Is a typical graph connected? Is it k-connected? Is the chromatic number at least k? Does almost every graph contain a triangle? Does it have diameter at most d? Is almost every graph Hamiltonian?

The greatest discovery of Erdős and Rényi was that many important properties of graphs appear quite suddenly. If we pick a function $M = M(n)$ then, in many cases, either almost every graph G_M has property Q or else almost every graph fails to have property Q. In this vague sense, we have a 0–1 law. The transition from a property being very unlikely to it being very likely is usually very swift. To make this more precise, consider a *monotone* (increasing) property Q, i.e. one for which a graph has Q whenever one of its subgraphs has Q. For many such properties there is a *threshold function* $M_0(n)$. If $M(n)$ grows somewhat slower than $M_0(n)$, then almost every G_M fails to have Q. If $M(n)$ grows somewhat faster than $M_0(n)$, then almost every G_M has the property Q. For example, $M_0(n) = \frac{1}{2}n \log n$ is a threshold function for connectedness in the following sense: if $\omega(n) \to \infty$, no matter how slowly, then almost every G is disconnected for $M(n) = \frac{1}{2}n(\log n - \omega(n))$ and almost every G is connected for $M(n) = \frac{1}{2}n(\log n + \omega(n))$.

As the proportion M/N of edges increases, where, as always, $N = \binom{n}{2}$ is the total number of possible edges, the shape of a typical graph G_M passes through several clearly identifiable stages, in which many of the important parameters of a typical graph are practically determined. When M is neither too small nor too large, then the most important property is that, for every fixed k, and k vertices in a typical graph have about the same number of neighbours. Thus a typical random graph is rather similar to an ideal regular graph whose automorphism group is transitive on small sets of vertices. Of course, there is no such non-trivial regular graph, and in many applications random graphs are used precisely because they approximate an ideal regular graph.

This book is the first systematic and extensive account of a substantial body of results from the theory of random graphs. Considerably shorter treatments of random graphs can be found in various parts of Erdős and Spencer (1974), in Chapter 8 of Marshall (1971), in Chapter 7 of Bollobás (1979*a*), and in the review papers by Grimmett (1980), Bollobás (1981*f*) and Karoński (1982). Despite over 750 references, several topics are not covered in any detail and we can make no claim to completeness. Perhaps the greatest omission is the extensive theory of random trees, about which the reader could get some idea by consulting the beautiful book by Moon (1970*a*) and the papers by Moon and Meir listed in the references. I might justify this particular omission because the tools used in its study are often those of enumeration rather than probability theory. However, here and in general, the choice of topics is mainly a reflection of my own interests.

The audience I had in mind when writing this book consisted mainly of research students and professional mathematicians. The volume should also be of interest to computer scientists. Random graphs are of ever increasing importance in this field and several of the sections have been written expressly for computer scientists.

The monograph was planned to be considerably shorter and was to be completed by early 1981. However, I soon realized that I had little control over the length which was dictated by the subject matter I intended to explore. Temptations to ease my load by adopting short-cuts have been rife but, for the reader's sake, I have tried to resist them. During its preparation I have given several courses on the material, notably for Part III for the University of Cambridge in 1980/81 and 1983/84. The first seven chapters were complete by the summer of 1982 and were circulated quite widely. I gave a series of lectures on these at the Waterloo Silver Jubilee Conference. These lectures were published only recently (1984*d*). I have written the book wherever I happened to be: the greatest part at Louisiana State University, Baton Rouge; a few chapters in Cambridge, Waterloo and São Paulo; and several sections in Tokyo and Budapest. I hope never again to travel with 200 lb of paper!

My main consideration in selecting and presenting the material was to write a book which I would like to read myself. In spite of this, the book is essentially self-contained, although familiarity with the basic concepts of graph theory and probability theory would be helpful. There is little doubt that many readers will use this monograph as a compendium of results. This is a pity, partly because a book suitable for that purpose could have been produced with much less effort than has gone into this volume, and also because proofs are often more important than results: not infrequently the reader will derive more benefit from knowing the methods used than from familiarity with the theorems. The list of contents describes fairly the material presented in the book.

The graph theoretic notation and terminology used in the book are standard. For undefined concepts and symbols the reader may consult Bollobás (1978*a*, 1979*a*).

The exercises at the end of each chapter vary greatly in importance and difficulty. Most are designed to clarify and complete certain points but a few are important results. The end of a proof, or its absence, is indicated by the symbol $\square$; the *floor* of x (i.e. the greatest integer less than, or equal to, x) is denoted by $\lfloor x \rfloor$; and the *ceiling* of x by $\lceil x \rceil$. With very few exceptions, the various parameters and random variables depend on the number n of vertices of graphs under consideration and the inequalities are only claimed to hold when n is sufficiently large. This is often stated but it is also implicit on many other occasions. The symbols $c_1, c_2, \ldots$ which appear without any explanation, are always independent of n. They may be absolute constants or may depend on other quantities which are independent of n. To assist the reader, it will often be stated which of these is the case.

It is a great pleasure to acknowledge the debt I owe to many people for their help. Several of my friends kindly read large sections of the manuscript. Andrew Thomason from Exeter, Istvan Simon from São Paulo, Masao Furuyama from Kyoto and Boris Pittel from Columbus were especially generous with their help. They corrected many errors and frequently improved the presentation. In addition, I benefited from the assistance of Andrew Barbour, Keith Carne, Geoff Eagleson, Alan Frieze and Jonathan Partington. Many research students, including Keith Ball, Graham Brightwell and Colin Wright, helped me find some of the mistakes; for the many which undoubtedly remain, I apologize. I am convinced that without the very generous help of Andrew Harris in using the computer the book would still be far from finished.

It was Paul Erdős and Alfréd Rényi who, just over 20 years ago, introduced me to the subject of random graphs. My active interest in the field was aroused by Paul Erdős some years later, whose love for it I found infectious; I am most grateful to him for firing my enthusiasm for the subject whose beauty has given me so much pleasure ever since.

Finally, I am especially grateful to Gabriella, my wife, for her constant support and encouragement. Her enthusiasm and patience survived even when mine failed.

Baton Rouge B.B.
December, 1984

Notation

$\beta_0(G)$	independence number	*page* 273, 282
$C(n, m)$	number of connected graphs	118
$\mathrm{cl}\,(G_n)$	clique number	282
$\mathrm{core}\,(G)$	core of G	150
$\mathrm{core}_k\,(G)$	k-core of G	150
$\delta(G)$	minimal degree	60
$\Delta(G)$	maximal degree	60
$(d_i)_1^n$	degree sequence	60
$\mathrm{diam}\,(G)$	diameter	251
E_M	expectation	34
E_p	expectation	35
E_r	rth factorial moment	3
$\mathrm{exc}\,(G)$	excess of G	148
$F = \mathbb{F}_q$	finite field	348
$G_M = G_{n,M}$	random graph	34
$G_p = G_{n,p}$	random graph	34
$G_r\text{-reg} = G_{n,r-\mathrm{reg}}$	random graph	50
$\widetilde{G} = (G_t)_0^N$	random graph process	42
$\widetilde{\mathcal{G}}$	space of graph processes	42
$\mathcal{G}(H; p)$	probability space of graphs	35
$\mathcal{G}(n; k\text{-out}) = \mathcal{G}_{k-\mathrm{out}}$	probability space of graphs	40
$\mathcal{G}(n, M) = \mathcal{G}_M$	probability space of graphs	34
$\mathcal{G}(n, p) = \mathcal{G}_p$	probability space of graphs	34
$\mathcal{G}(n, r\text{-reg}) = \mathcal{G}_{r-\mathrm{reg}}$	probability space of graphs	50
$\mathcal{G}\{n, P(\mathrm{edge})\}$	probability space of graphs	34
$\ker\,(G)$	kernel of G	150
$L_M = L_{n\,M}$	number of labelled graphs	229
$N(\mu, \sigma)$	normal distribution	9

P_k	property	44
P_λ	Poisson distribution	8
P_M	probability in $\mathscr{G}(n, M)$	34
P_p	probability in $\mathscr{G}(n, P)$	35
P_q	Paley graph	357
Q^n	n-dimensional cube	383
Q^n_p	random subgraph of Q^n	383
$R(s)$, $R(s, t)$	Ramsey numbers	320
$S_{n,p}$	binomial r.v.	5
$\tau = \tau_Q = \tau_Q(\widetilde{G}) = \tau(\widetilde{G}; Q)$	hitting time	42
$t(c)$	function concerning the number of vertices on tree components	102
$U_M = U_{n,M}$	number of unlabelled graphs	229
$u(c)$	function concerning the number of tree components	109
$\chi(G)$	chromatic number	296
$X_n \overset{d}{\to} X$	convergence in distribution	2
$\omega(n)$	function tending to ∞	43