LINEAR WATER WAVES

A Mathematical Approach

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Introduction: Basic Theory of Surface Waves

Here we give a brief account of physical assumptions (first section) and the mathematical approximation (second section) used for developing a mathematical model of water waves. The resulting linear boundary value problems are formulated in the third and fourth sections for the wave–body interaction and ship waves, respectively.

Mathematical Formulation

Conventions

Water waves (the terms *surface waves* and *gravity waves* are also in use) are created normally by a gravitational force in the presence of a free surface along which the pressure is constant. There are two ways to describe these waves mathematically. It is possible to trace the paths of individual particles (a Lagrangian description), but in this book an alternative form of equations (usually referred to as Eulerian) is adopted. The motion is determined by the velocity field in the domain occupied by water at every moment of the time t.

Water is assumed to occupy a certain domain W bounded by one or more moving or fixed surfaces that separate water from some other medium. Actually we consider boundaries of two types: the above-mentioned free surface separating water from the atmosphere, and rigid surfaces including the bottom and surfaces of bodies floating in and/or beneath the free surface.

It is convenient to use rectangular coordinates (x_1, x_2, y) with origin in the free surface at rest (which usually coincides with the mean free surface), and with the y axis directed opposite to the acceleration caused by gravity. For the sake of brevity we will write x instead of (x_1, x_2). This has the obvious advantage that two- and three-dimensional problems can be treated simultaneously, where it is possible. *Two-dimensional problems* form an important class of problems considering water motions that are the same in every plane orthogonal to a certain direction. Subscripts will be used to denote (partial) derivatives, for example:

$$u_t = \frac{\partial u}{\partial t}, \quad u_y = \frac{\partial u}{\partial y}, \quad u_{x_i} = \frac{\partial u}{\partial x_i}, \quad i = 1, 2.$$

When this notation is inconvenient, we will apply the following one:

$$\partial_t u, \, \partial_y u, \, \partial_{x_i} u, \ldots .$$

As usual, $\nabla u = (u_{x_1}, u_{x_2}, u_y)$, and the horizontal component of ∇ will be denoted by ∇_x, that is, $\nabla_x u = (u_{x_1}, u_{x_2}, 0)$. Clearly, $\nabla u = (u_x, u_y)$ and $\nabla_x u = (u_x, 0)$ in two-dimensional problems.

In several chapters, in particular in those concerned with the forward motion of a body, we use (x, z) instead of (x_1, x_2).

Equations of Motion and Boundary Conditions

In the Eulerian formulation one seeks the velocity vector $\mathbf{v}$, the pressure p, and the fluid density ρ as functions of $(x, y) \in \bar{W}$ and $t \geq t_0$, where t_0 denotes a certain initial moment. Assuming the fluid to be inviscid without surface tension, one obtains the equations of motion from conservation laws (for details see, for example, books by Lamb [179], Le Méhauté [186], and Stoker [312]).

The conservation of mass implies the continuity equation

$$\rho_t + \nabla \cdot (\rho \mathbf{v}) = 0 \quad \text{in } W.$$

Under the assumption that the fluid is incompressible (which is usual in the water-wave theory), the last equation becomes

$$\nabla \cdot \mathbf{v} = 0 \quad \text{in } W. \tag{I.1}$$

The conservation of momentum in inviscid fluid leads to the so-called Euler equations. Taking into account the gravity force, one can write these three (or two) equations in the following vector form:

$$\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = -\rho^{-1} \nabla p + \mathbf{g}. \tag{I.2}$$

Here $\mathbf{g}$ is the vector of the gravity force having zero horizontal components and the vertical one equal to $-g$, where g denotes the acceleration caused by gravity.

An irrotational character of motion is another usual assumption in the water-wave theory; that is,

$$\nabla \times \mathbf{v} = 0 \quad \text{in } W.$$

Note that one can prove that the motion is irrotational if it has this property at the initial moment (see, for example, books by Lamb [179] and Stoker [312] for the proof of this assertion known as the Helmholtz theorem). The last equation guarantees the existence of a velocity potential ϕ, so that

$$\mathbf{v} = \nabla\phi \quad \text{in } \bar{W}. \tag{I.3}$$

This is obvious for simply connected domains; otherwise (for example, when one considers a two-dimensional problem for a totally immersed body), the so-called no flow condition, see (I.8) below, should be taken into account.

From (I.1) and (I.3) one obtains the Laplace equation

$$\nabla^2\phi = 0 \quad \text{in } W. \tag{I.4}$$

This greatly facilitates the theory but, in general, solutions of (I.4) do not manifest wave character. Waves are created by the boundary conditions on the free surface.

Let $y = \eta(x, t)$ be the equation of the free surface valid for $x \in F$, where F is a union of some domains (generally depending on t) in $\mathbb{R}^n$, with $n = 1, 2$. The pressure is prescribed to be equal to the constant atmospheric pressure p_0 on $y = \eta(x, t)$, and the surface tension is neglected. From (I.2) and (I.3) one immediately obtains Bernoulli's equation,

$$\phi_t + |\nabla\phi|^2/2 = -\rho^{-1}p - gy + C \quad \text{in } \bar{W}, \tag{I.5}$$

where C is a function of t alone. Indeed, applying ∇ to both sides in (I.5) and using (I.2) and (I.3), one obtains $\nabla C = 0$. Then, by changing ϕ by a suitable additive function of t, one can convert C into a constant having, for example, the value

$$C = \rho^{-1}p_0. \tag{I.6}$$

Now (I.5) gives the *dynamic boundary condition* on the free surface:

$$g\eta + \phi_t + |\nabla\phi|^2/2 = 0 \quad \text{for } y = \eta(x, t), x \in F. \tag{I.7}$$

Another boundary condition holds on every "physical" surface $\mathcal{S}$ bounding the fluid domain W and expresses the kinematic property that there is no transfer of matter across $\mathcal{S}$. Let $s(x, y, t) = 0$ be the equation of $\mathcal{S}$; then

$$\mathrm{d}s/\mathrm{d}t = \mathbf{v}\cdot\nabla s + s_t = 0 \quad \text{on } \mathcal{S}. \tag{I.8}$$

Under assumption (I.3) this takes the form of

$$\frac{\partial\phi}{\partial n} = -\frac{s_t}{|\nabla s|} = v_n \quad \text{on } \mathcal{S}, \tag{I.9}$$

where v_n denotes the *normal velocity* of $\mathcal{S}$. Thus the *kinematic boundary condition* (I.9) means that the normal velocity of particles is continuous across a physical boundary.

On the fixed part of $\mathcal{S}$, (I.9) takes the form of

$$\partial\phi/\partial n = 0. \tag{I.10}$$

On the free surface, condition (I.8), written as follows,

$$\eta_t + \nabla_x\phi \cdot \nabla_x\eta - \phi_y = 0 \quad \text{for } y = \eta(x,t),\, x \in F, \tag{I.11}$$

complements the dynamic condition (I.7). Thus, in the present approach, two nonlinear conditions (I.7) and (I.11) on the unknown boundary are responsible for waves, which constitutes the main characteristic feature of water–surface wave theory.

This brief account of governing equations can be summarized as follows.

In the water-wave problem one seeks the velocity potential $\phi(x, y, t)$ and the free surface elevation $\eta(x, t)$ satisfying (I.4), (I.7), (I.9), *and* (I.11). *The initial values of ϕ and η should also be prescribed, as well as the conditions at infinity (for unbounded W) to complete the problem, which is known as the Cauchy–Poisson problem.*

Energy and Its Flow

Let W_0 be a subdomain of W bounded by a "geometric" surface ∂W_0 that may not be related to physical obstacles and that is permitted to vary in time independently of moving water unlike "physical" surfaces described below. Let $s_0(x, y, t) = 0$ be the equation of ∂W_0. The total energy contained in W_0 consists of kinetic and potential components and is given by

$$E = \rho \int_{W_0} [gy + |\nabla\phi|^2/2]\,\mathrm{d}x\mathrm{d}y. \tag{I.12}$$

The first term related to the vertical displacement of a water particle corresponds to the potential energy, whereas the second one gives the kinetic energy that is proportional to the velocity squared. Using (I.5) and (I.6), one can write this in the form of

$$E = -\rho \int_{W_0} (\rho\phi_t + p - p_0)\,\mathrm{d}x\mathrm{d}y.$$

Differentiating (I.12) with respect to t we get

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \rho \int_{W_0} \nabla\phi \cdot \nabla\phi_t\,\mathrm{d}x\mathrm{d}y + \int_{\partial W_0} \frac{s_{0t}}{|\nabla s_0|}(\rho\phi_t + p - p_0)\,\mathrm{d}S.$$

Green's theorem applied to the first integral here leads to

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_{\partial W_0} \left[\rho\phi_t \left(\frac{\partial\phi}{\partial n} - v_n \right) - (p - p_0)v_n \right] \mathrm{d}S, \tag{I.13}$$

where (I.4) is taken into account and v_n denotes the normal velocity of ∂W_0. Hence the integrand in (I.13) is the rate of energy flow from W_0 through ∂W_0 taken per units of time and area. The velocity of energy propagation is known as the group velocity. However, it does not play any significant role in considerations presented in this book, and we restrict ourselves to references to works of Stoker [312] and Wehausen and Laitone [354], where further details can be found.

If a portion of ∂W_0 is a fixed geometric surface, then $v_n = 0$ on this portion; the rate of energy flow is given by $-\rho\phi_t(\partial\phi/\partial n)$.

If a portion of ∂W_0 is a "physical" boundary that is not penetrable by water particles, then (I.9) shows that the integrand in (I.13) is equal to $(p_0 - p)v_n$. Therefore, there is no energy flow through this portion of ∂W_0 if either of two factors vanishes. In particular, this is true for the free surface ($p = p_0$) and for the bottom ($v_n = 0$).

Linearized Unsteady Problem

Linearization: Its Applicability and Justification

About 50 years ago, John [125] assessed the problem formulated at the end of the subsection on equations of motion and boundary conditions as follows:

> In this generality little can be done either toward a discussion of the motion or toward an explicit solution of the equations. The difficulties arising from the fact that ϕ is a solution of the potential equation determined by non-linear boundary conditions on a variable boundary are considerable, and have only been overcome in the special cases of permanent waves treated by Levi-Civita and Struik.

Here works [194, 314] by Levi-Civita and Struik, respectively, are cited (see also Nekrasov's work [261]).

Since then a large number of papers has been published and great progress has been achieved in the mathematical treatment of nonlinear water-wave problems (we list only a few works: Debnath [46], Kirchgässner [138], Olver [272] and Ovsyannikov et al. [273], where further references can be found). However, all rigorous results in this direction are concerned with water waves in the absence of floating bodies, and the present state of the art for the *nonlinear problem for floating bodies* is the same as 50 years ago. Of course, a substantial body of numerical results treating different aspects of the nonlinear problem has emerged during the past three decades, but this approach is beyond our scope.

To be in a position to describe water waves in the presence of bodies, the equations should be approximated by more tractable ones. The usual and rather reasonable simplification consists of a *linearization* of the problem under certain assumptions concerning the motion of a floating body. An example of such assumptions (there are other ones leading to the same conclusions) suggests that a body's motion near the equilibrium position is so small that it produces only waves having a small amplitude and a small wavelength. There are three characteristic geometric parameters:

1. A typical value of the wave height H.
2. A typical wavelength L.
3. The water depth D.

They give three characteristic quotients: H/L, H/D, and L/D. The relative importance of these quotients is different in different situations. Nevertheless, it was found (see, for example, Le Méhauté [186], Sections 15-2 and 15-3) that if

$$\frac{H}{D} \ll 1 \quad \text{and} \; \frac{H}{L}\left(\frac{L}{D}\right)^3 \ll 1,$$

then the linearization can be justified by some heuristic considerations. The last parameter $(H/L)(L/D)^3 = (H/D)(L/D)^2$ is usually referred to as Ursell's number. Its role in a classification of water waves is presented in detail by Le Méhauté [186], Section 15-2. Further results treating the problem of linearization can be found in the paper [22] by Beale, Hou, and Lowengrub.

The linearized theory leads to results that are in a rather good agreement with experiments and observations. During the 1940s and 1950s, a substantial work in this direction was carried out by Ursell and his coauthors. Thus Barber and Ursell [19] discovered a good agreement between predictions of the linear theory for group velocity and values resulting from observations, and Ursell [325] demonstrated the same for frequencies. Some experiments were carried out by Dean, Ursell, and Yu [45] and by Ursell and Yu [343], and in a certain range of wave steepnesses a very close agreement was obtained between the measured wave amplitude (up to some corrections inevitable in an experiment) and theoretical predictions made on the base of the linear problem.

Furthermore, there is mathematical evidence that the linearized problem provides an approximation to the nonlinear one. For the Cauchy–Poisson problem describing waves in a water layer caused by prescribed initial conditions, the linear approximation is justified rigorously by Nalimov (see the book by Ovsyannikov et al. [273], Chapter 3). More precisely, under the assumption that the undisturbed water occupies a layer of constant depth, the following are proved:

1. *The nonlinear problem is solvable for sufficiently small values of the linearization parameter.*
2. *As this parameter tends to zero, solutions of the nonlinear problem do converge to the solution of the linearized problem in the norm of some suitable function space.*

Equations for Small Amplitude Waves

A formal perturbation procedure leading to a sequence of linear problems can be developed as follows. Let us assume that the velocity potential ϕ and the free surface elevation η admit expansions with respect to a certain small parameter ϵ:

$$\phi(x, y, t) = \epsilon\phi^{(1)}(x, y, t) + \epsilon^2\phi^{(2)}(x, y, t) + \epsilon^3\phi^{(3)}(x, y, t) + \cdots, \quad \text{(I.14)}$$

$$\eta(x, t) = \eta^{(0)}(x, t) + \epsilon\eta^{(1)}(x, t) + \epsilon^2\eta^{(2)}(x, t) + \cdots, \quad \text{(I.15)}$$

where $\phi^{(1)}$, $\phi^{(2)}, \ldots,$ $\eta^{(0)}$, $\eta^{(1)}, \ldots,$ and all their derivatives are bounded. Consequently, the velocities of water particles are supposed to be small (proportional to ϵ), and $\epsilon = 0$ corresponds to water permanently at rest.

Substituting (I.14) into (I.4) gives

$$\nabla^2\phi^{(k)} = 0 \quad \text{in } W, \quad k = 1, 2, \ldots. \quad \text{(I.16)}$$

Furthermore, $\eta^{(0)}$ describing the free surface at rest cannot depend on t. When the expansions for ϕ and η are substituted into the Bernoulli boundary condition (I.7) and grouped according to powers of ϵ, one obtains

$$\eta^{(0)} \equiv 0 \quad \text{for } x \in F.$$

This and Taylor's expansion of $\phi[x, \eta(x, t), t]$ in powers of ϵ yield the following for orders higher than zero:

$$\phi_t^{(1)} + g\eta^{(1)} = 0 \quad \text{for } y = 0, x \in F, \quad \text{(I.17)}$$

$$\phi_t^{(2)} + g\eta^{(2)} = -\eta^{(1)}\phi_{ty}^{(1)} - \left|\nabla\phi^{(1)}\right|^2/2 \quad \text{for } y = 0,\ x \in F, \quad \text{(I.18)}$$

and so on; that is, all these conditions hold on the mean position of the free surface at rest.

Similarly, the kinematic condition (I.11) leads to

$$\phi_y^{(1)} - \eta_t^{(1)} = 0 \quad \text{for } y = 0, x \in F, \tag{I.19}$$

$$\phi_y^{(2)} - \eta_t^{(2)} = -\eta^{(1)}\phi_{yy}^{(1)} + \nabla_x\phi^{(1)} \cdot \nabla_x\eta^{(1)} \quad \text{for } y = 0, x \in F, \tag{I.20}$$

and so on. Eliminating $\eta^{(1)}$ between (I.17) and (I.19), one finds the classical first-order linear free-surface condition:

$$\phi_{tt}^{(1)} + g\phi_y^{(1)} = 0 \quad \text{for } y = 0, x \in F. \tag{I.21}$$

In the same way, one obtains from (I.18) and (I.20) the following:

$$\phi_{tt}^{(2)} + g\phi_y^{(2)} = -\phi_t^{(1)}\nabla_x^2\phi^{(1)} - \frac{1}{g^2}\left[\phi_t^{(1)}\phi_{ttt}^{(1)} + \left|\nabla_x\phi^{(1)}\right|^2\right]_t \quad \text{for } y = 0, x \in F.$$

Further free-surface conditions can be obtained for terms in (I.14) having higher orders in ϵ. All these conditions have the same operator in the left-hand side, and the right-hand term depends nonlinearly on terms of smaller orders. It is worth mentioning that all of the high-order problems are formulated in the same domain W occupied by water at rest. In particular, the free-surface boundary conditions are imposed at $\{y = 0, x \in F\}$.

Boundary Condition on an Immersed Rigid Surface

First, we note that the homogeneous Neumann condition (I.10) is linear on fixed surfaces. Hence, this condition is true for $\phi^{(k)}$, $k = 1, 2, \ldots$. The situation reverses for the inhomogeneous Neumann condition (I.9) on a moving surface $\mathcal{S}$, which can be subjected, for example, to a prescribed motion or freely floating. The problem of a body freely floating near its equilibrium position will not be treated in the book (for linearization of this problem see John's paper [125]). We restrict ourselves to the linearization of (I.9) for $\mathcal{S} = S(t, \epsilon)$ undergoing a given small amplitude motion near an equilibrium position S, that is, when $S(t, \epsilon)$ tends to S as $\epsilon \to 0$.

It is convenient to carry out the linearization locally. Let us consider a neighborhood of $(x^{(0)}, y^{(0)}) \in S$, where the surface is given explicitly in local Cartesian coordinates (ξ, ζ), where in the three-dimensional case $\xi = (\xi_1, \xi_2)$, having an origin at $(x^{(0)}, y^{(0)})$ and the ζ axis directed into water normally to S. Let $\zeta = \zeta^{(0)}(\xi)$ be the equation of S, and $S(t, \epsilon)$ be given by $\zeta = \zeta(\xi, t, \epsilon)$, where

$$\zeta(\xi, t, \epsilon) = \zeta^{(0)}(\xi) + \epsilon\zeta^{(1)}(\xi, t) + \epsilon^2\zeta^{(2)}(\xi, t) + \cdots. \tag{I.22}$$

After substituting (I.14) and $s = \zeta - \zeta(\xi, t, \epsilon)$ into (I.8), we use (I.3), (I.22), and Taylor's expansion in the same way as in the subsection on equations for small amplitude waves. This gives the following first-order equation:

$$\phi_\zeta^{(1)}\big(\xi, \zeta^{(0)}, t\big) - \nabla_\xi \phi^{(1)}\big(\xi, \zeta^{(0)}, t\big) \cdot \nabla_\xi \zeta^{(0)}(\xi) = \zeta_t^{(1)}(\xi, t),$$

which implies the linearized boundary condition:

$$\partial \phi^{(1)}/\partial n = v_n^{(1)} \quad \text{on } S, \tag{I.23}$$

where

$$v_n^{(1)} = \zeta_t^{(1)} \big/ \big[1 + \big|\nabla_\xi \zeta^{(0)}\big|^2\big]^{1/2}$$

is the first-order approximation of the normal velocity of $S(t, \epsilon)$.

The second-order boundary condition on S has the form

$$\frac{\partial \phi^{(2)}}{\partial n} = \frac{\zeta_t^{(2)}}{\big[1 + \big|\nabla_\xi \zeta^{(0)}\big|^2\big]^{1/2}} - \zeta^{(1)} \frac{\partial^2 \phi^{(1)}}{\partial n^2} - \left[\frac{1 + \big|\nabla_\xi \zeta^{(1)}\big|^2}{1 + \big|\nabla_\xi \zeta^{(0)}\big|^2}\right]^{1/2} \frac{\partial \phi^{(1)}}{\partial n^{(1)}},$$

where $\partial \phi^{(1)}/\partial n^{(1)}$ is the derivative in the direction of normal to $\zeta = \zeta^{(1)}(\xi, t)$ calculated on S. In addition, further conditions on S of the Neumann type can be obtained for terms of higher order in ϵ.

Thus, all $\phi^{(k)}$ satisfy the same linear boundary value problem with different right-hand-side terms in conditions on the free surface at rest and on the equilibrium surfaces of immersed bodies. These right-hand-side terms depend on solutions obtained on previous steps. Solving these problems successively, beginning with problems (I.16), (I.21), and (I.23) complemented by some initial conditions, one can, generally speaking, find a solution to the nonlinear problem in the form of (I.14) and (I.15). However, this procedure is not justified mathematically up to the present time. Therefore, in this book we restrict ourselves to the first-order approximation, which in its own right gives rise to an extensive mathematical theory. Investigations in this field are far from being exhausted.

We conclude this subsection by summarizing the boundary value problem for *the first-order velocity potential* $\phi^{(1)}(x, y, t)$. *It is defined in* W *occupied by water at rest with a boundary consisting of the free surface* F, *the bottom* B, *and the wetted surface of immersed bodies* S, *and it must satisfy*

$$\nabla^2 \phi^{(1)} = 0 \quad \text{in } W, \tag{I.24}$$

$$\phi_{tt}^{(1)} + g\phi^{(1)} = 0 \quad \text{for } y = 0, x \in F, \tag{I.25}$$

$$\partial \phi^{(1)}/\partial n = v_n^{(1)} \quad \text{on } S, \tag{I.26}$$

$$\partial \phi^{(1)}/\partial n = 0 \quad \text{on } B, \tag{I.27}$$

$$\phi^{(1)}(x, 0, 0) = \phi_0(x) \quad \text{and } \phi_t^{(1)}(x, 0, 0) = -g\eta_0(x), \tag{I.28}$$

where ϕ_0, $v_n^{(1)}$, *and* η_0 *are given functions, and* $\eta_0(x) = \eta^{(1)}(x, 0)$*; see* (I.17). *Then*

$$\eta^{(1)}(x, t) = -g^{-1}\phi_t^{(1)}(x, 0, t)$$

gives the first-order approximation for the elevation of the free surface.

In conclusion of the section, it should be mentioned that for the case of a rigid body freely floating near an equilibrium position, a linearized system of coupled equations was proposed by John [125]. This system was investigated by John [126], Beale [21], and Licht [197, 200]. Another coupled initial-boundary value problem dealing with a fixed elastic body immersed in water was considered by Licht [198, 199].

Linear Time-Harmonic Waves (the Water-Wave Problem)

Separation of the t variable

We pointed out in the preface that this book is concerned with the steady-state problem of radiation and scattering of water waves by bodies floating in and/or beneath the free surface, assuming all motions to be simple harmonic in the time. The corresponding radian frequency is denoted by ω. Thus, the right-hand-side term in (I.23) is

$$v_n^{(1)} = \mathrm{Re}\{e^{-i\omega t} f\} \quad \text{on } S, \tag{I.29}$$

where f is a complex function independent of t, and the first-order velocity potential $\phi^{(1)}$ can then be written in the form

$$\phi^{(1)}(x, y, t) = \mathrm{Re}\{e^{-i\omega t} u(x, y)\}. \tag{I.30}$$

The latter assumption is justified by the so-called limiting amplitude principle, which is concerned with the large-time behavior of a solution to the initial-boundary value problem having (I.29) as the right-hand-side term. According to this principle, such a solution tends to the potential (I.30) as $t \to \infty$, and u satisfies a steady-state problem. The limiting amplitude principle has general applicability in the theory of wave motions, and its particular form for water waves was proved by Vullierme-Ledard [349]. Thus the problem of our interest describes waves developing at large time from time-periodic disturbances.

A complex function u in (I.30) is also referred to as velocity potential (this does not lead to confusion, because it will always be clear what kind of time dependence is considered in one part of the book or another). We recall that u is defined in the fixed domain W occupied by water at rest outside any bodies present. The boundary ∂W consists of three disjoint sets: (i) S, which is the union of the wetted surfaces of bodies in equilibrium; (ii) F, denoting the free surface at rest that is the part of $y = 0$ outside all the bodies; and (iii) B, which denotes the bottom positioned below $F \cup S$. Sometimes we will consider W unbounded below and corresponding to infinitely deep water. In this case $\partial W = F \cup S$.

Substituting (I.29) and (I.30) into (I.24)–(I.27) gives the boundary value problem for u:

$$\nabla^2 u = 0 \quad \text{in } W, \tag{I.31}$$

$$u_y - \nu u = 0 \quad \text{on } F, \tag{I.32}$$

$$\partial u / \partial n = f \quad \text{on } S, \tag{I.33}$$

$$\partial u / \partial n = 0 \quad \text{on } B, \tag{I.34}$$

where $\nu = \omega^2 / g$. Throughout the book a normal n to a surface always directs *into* the water domain W.

For deep water ($B = \emptyset$), condition (I.34) should be replaced by the following one:

$$\sup_{(x,y) \in W} |u(x, y)| \leq \text{const} < \infty. \tag{I.35}$$

Despite the fact that this condition has no direct hydrodynamic meaning, we impose it because it is essential for certain proofs in what follows. Besides, (I.35) implies the following natural behavior of the velocity filed (see Subsection 1.1.1.1 for the proof):

$$|\nabla u| \to 0 \quad \text{as } y \to -\infty; \tag{I.36}$$

that is, the water motion decays with depth. Conditions at infinity that are similar to the last two conditions are usually imposed in the boundary value problems for the Laplacian in domains exterior to a compact set in $\mathbb{R}^2$ and $\mathbb{R}^3$. A natural requirement that a solution to (I.31)–(I.35) should be unique also imposes a certain restriction on the behavior of u as $|x| \to \infty$. We discuss conditions providing uniqueness in the subsection after the following one.

Examples

Let us consider some simple examples of waves existing in the absence of bodies. The corresponding potentials can be easily obtained by separation of variables.

For a layer W of constant depth d, $F = \{x \in \mathbb{R}^2,\ y = 0\}$ and $B = \{x \in \mathbb{R}^2,\ y = -d\}$ are the free surface and bottom, respectively. *A plane progressive* *wave* propagating in the direction of a wave vector $\mathbf{k} = (k_1, k_2)$ has the following velocity potential:

$$\mathrm{Re}\{A \exp[i(\mathbf{k} \cdot x - \omega t)]\} \cosh k_0(y + d). \tag{I.37}$$

Here A is an arbitrary complex constant, $k_0 = |\mathbf{k}|$, and the following relationship,

$$\nu = \omega^2/g = k_0 \tanh k_0 d, \tag{I.38}$$

holds between ω and k_0. Tending d to infinity, we note that k_0 becomes equal to ν and instead of (I.37) we have

$$\mathrm{Re}\{A \exp[i(\mathbf{k} \cdot x - \omega t)]\} e^{\nu y}$$

for the velocity potential of plane progressive wave in deep water.

A sum of two potentials (I.37) corresponding to identical progressive waves propagating in opposite directions gives a standing wave. Putting $\exp \nu y$ instead of $\cosh k_0(y + d)$ in (I.37) and omitting $\tanh k_0 d$ in (I.38), one gets the potential of a progressive wave in deep water.

A standing cylindrical wave in a water layer of depth d has the following potential:

$$w_{\mathrm{st}}(x, y) \cos \omega t, \quad \text{where } w_{\mathrm{st}}(x, y) = C_1 \cosh k_0(y + d) J_0(k_0|x|),$$

where k_0 is defined by (I.38), C_1 is a real constant, and J_0 denotes the Bessel function of order zero. The same manipulation as above gives the standing wave in deep water.

A cylindrical wave having an arbitrary phase at infinity may be obtained as a combination of w_{st} and a similar potential with J_0 replaced by Y_0, which is another solution of Bessel's equation. This allows one to construct a potential of outgoing wave as follows:

$$\mathrm{Re}\{e^{-i\omega t} w_{\mathrm{out}}(x, y)\}, \quad \text{where } w_{\mathrm{out}}(x, y) = C_2 \cosh k_0(y + d) H_0^{(1)}(k_0|x|),$$

where k_0 is defined by (I.38), C_2 is a complex constant, and $H_0^{(1)}$ denotes the first Hankel function of order zero. Outgoing behavior of this wave becomes clear from the asymptotic formula (see handbooks by Abramowitz & Stegun [1], and Gradshteyn and Ryzhik [96]):

$$H_0^{(1)}(k_0|x|) = \left(\frac{2}{\pi k_0|x|}\right)^{1/2} e^{i(k_0|x| - \pi/4)}[1 + O(|x|^{-1})] \quad \text{as } |x| \to \infty.$$

Therefore, wave w_{out} behaves at large distances like a radially outgoing progressive wave, but it is singular on the axis $|x| = 0$. This is natural from a physical point of view, because outgoing waves should be radiated by a certain disturbance. In the case under consideration, the wave is produced by sources distributed with a suitable density over $\{|x| = 0, -d < y < 0\}$. Replacing $H_0^{(1)}$ in w_{out} by the second Hankel function, $H_0^{(2)}$, one obtains an incoming wave.

Radiation Conditions

Examples in the previous subsection demonstrate that problem (I.31)–(I.34) should be complemented by an appropriate condition as $|x| \to \infty$ to avoid non-uniqueness of the solution, which follows from the fact that there are infinitely many solutions of the form of (I.37). On the other hand, the energy dissipates when waves are radiated or scattered; that is, there exists a flow of energy to infinity. On the contrary, there is no such a flow for standing waves and no net flow for progressive waves. Since we are going to describe radiation and scattering phenomena, a condition should be introduced for eliminating waves having no flow of energy to infinity. For this purpose a mathematical expression is used known as a *radiation condition*. To formulate this condition we have to specify the geometry of the water domain at infinity.

Let W be an $(m+1)$-dimensional domain ($m = 1, 2$), which at infinity coincides with the layer $\{x \in \mathbb{R}^m, -d < y < 0\}$, where $0 < d \leq \infty$. We say that u satisfies the radiation condition of the Sommerfeld type if

$$u_{|x|} - ik_0 u = \sigma(y) o\left[|x|^{(1-m)/2}\right] \quad \text{as } |x| \to \infty \text{ uniformly in } y, \theta. \tag{I.39}$$

Here $\sigma(y) = (1+|y|)^{-m}$ if $d = \infty$, $\sigma(y) = 1$ if $d < \infty$, k_0 is defined by (I.38) for $d < \infty$, and $k_0 = \nu$ for $d = \infty$, and $\theta \in [0, 2\pi)$ is polar angle in the plane $\{y = 0\}$. Uniformity in θ should be imposed only for the three-dimensional problem ($m = 2$).

Let us show that (I.39) guarantees dissipation of energy. For the sake of simplicity we assume that $d < \infty$. By $\mathcal{C}_r$ we denote a cylindrical surface $W \cap \{|x| = r\}$ contained inside W. By (I.13) the average energy flow to infinity through $\mathcal{C}_r$ over one period of oscillations is equal to

$$\mathcal{F}_r = -\frac{\rho\omega}{2\pi} \int_0^{2\pi/\omega} \mathrm{d}t \int_{\mathcal{C}_r} \frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial |x|} \, \mathrm{d}S.$$

Substituting (I.30) and taking into account that

$$\int_0^{2\pi/\omega} e^{\pm 2i\omega t} \, \mathrm{d}t = 0,$$

one finds that

$$\begin{aligned}\mathcal{F}_r &= -\frac{\rho\omega^2}{8\pi} \int_0^{2\pi/\omega} \mathrm{d}t \int_{\mathcal{C}_r} (ie^{i\omega t}\bar{u} - ie^{-i\omega t}u)\big(e^{-i\omega t}u_{|x|} + e^{i\omega t}\bar{u}_{|x|}\big) \, \mathrm{d}S \\ &= -\frac{\rho\omega}{4\pi} \int_{\mathcal{C}_r} \big(i\bar{u}u_{|x|} - iu\bar{u}_{|x|}\big) \, \mathrm{d}S = \frac{\rho\omega}{2} \operatorname{Im} \int_{\mathcal{C}_r} \bar{u}\, u_{|x|} \, \mathrm{d}S. \end{aligned} \tag{I.40}$$

This can be written as follows:

$$\mathcal{F}_r = \frac{\rho\omega}{4k_0}\left\{\int_{\mathcal{C}_r}\left(\left|u_{|x|}\right|^2 + k_0^2|u|^2\right)\mathrm{d}S - \int_{\mathcal{C}_r}\left|u_{|x|} - ik_0u\right|^2\mathrm{d}S\right\}. \quad \text{(I.41)}$$

Moreover, $\mathcal{F}_r$ does not depend on r when the obstacle surface S lies inside the cylinder $\{|x| = r\}$, which can be proved as follows.

By W_r and F_r we denote $W \cap \{|x| < r\}$ and $F \cap \{|x| < r\}$, respectively. Let us multiply (I.31) by $\bar{u}$ and integrate the result over W_r. Then applying the divergence theorem we obtain

$$\int_{W_r} |\nabla u|^2\,\mathrm{d}x\mathrm{d}y = -\int_{\partial W_r} \bar{u}\,\frac{\partial u}{\partial n}\,\mathrm{d}S,$$

where n is directed into W_r. Using (I.32) and (I.34) we get

$$\int_{W_r} |\nabla u|^2\,\mathrm{d}x\mathrm{d}y = \nu\int_{F_r} |u|^2\,\mathrm{d}x + \int_{\mathcal{C}_r} \bar{u}\,u_{|x|}\,\mathrm{d}S - \int_S \bar{u}\,\frac{\partial u}{\partial n}\,\mathrm{d}S.$$

Comparing this with (I.40) we find that

$$\mathcal{F}_r = \frac{\rho\omega}{2}\,\mathrm{Im}\int_S \bar{u}\,\frac{\partial u}{\partial n}\,\mathrm{d}S$$

is independent of r.

This fact yields that $\mathcal{F}_r \geq 0$ because (I.39) implies that the last integral in (I.41) tends to zero as $r \to \infty$.

The crucial point in the proof that $\mathcal{F}_r \geq 0$ is equality (I.41). It suggests that (I.39) can be replaced by a "weaker" radiation condition of the Rellich type,

$$\int_{\mathcal{C}_r}\left|u_{|x|} - ik_0u\right|^2\mathrm{d}S = o(1) \quad \text{as } r \to \infty. \quad \text{(I.42)}$$

Actually, (I.39) and (I.42) are equivalent (see the Subsection 1.3.2).

So, in what follows we consider problem (I.31)–(I.34) complemented by either (I.39) or (I.42). In various papers this problem appears under different names: the floating-body problem, the sea-keeping problem, the wave–body interaction problem, the water-wave radiation (scattering) problem, and so on. In what follows we use the simplest name: the water-wave problem.

Other Time-Harmonic Problems

In conclusion of the present section, we mention some boundary value problems that couple time-harmonic water waves with oscillations in other media. Hazard and Lenoir [113] considered scattering of an incident water wave by an elastic body immersed in water (the corresponding initial-boundary value problem was treated by Licht [199]). A linearized model of water-wave motion in a porous structure was proposed by Sollitt and Cross [308] for describing the interaction of water waves with rubble-mound breakwaters. This model was investigated by McIver [237], where further references are given. The most recent coupled problem was advanced by Pinkster [289] and investigated by Newman [267]. It is concerned with acoustic waves in a bounded air chamber placed on the free surface of water and open from below for interaction with water waves.

Linear Ship Waves on Calm Water (the Neumann–Kelvin Problem)

Separation of the t variable

Here we turn to waves created by a rigid body moving uniformly with constant velocity U on a calm water of constant depth d. It is convenient to denote the horizontal coordinates by (x, z) instead of (x_1, x_2). We assume (without loss of generality) that the motion is along the x axis of a fixed coordinate system. Moreover, we suppose waves to be steady with respect to a moving coordinate system attached to the body, or, in other words, one may speak about a uniform running flow about the body. The flow carries steady waves downstream (from the body to $x = -\infty$), so we set the following in (I.24)–(I.27):

$$\phi^{(1)}(x, y, z, t) = u(x - Ut, y, z), \tag{I.43}$$

where the (x, y, z)-coordinate system is fixed. Using the same notation (x, y, z) for the system attached to the body (since we use only these coordinates in what follows, this does not lead to any confusion), we see that the velocity potential $u(x, y, z)$ is defined in a fixed domain W occupied by water at rest outside the body's surface S. Since the water depth is constant, W is bounded below by $y = -d$ ($d \in (0, +\infty]$ and $d = +\infty$ for deep water), and we assume that S has no common points with this plane when $d < +\infty$. As in the third major section (the water-wave problem), we denote by F the free surface at rest that is the part of $y = 0$ outside the body.

Substituting (I.43) into (I.24)–(I.27), one obtains the following for u:

$$\nabla^2 u = 0 \quad \text{in } W, \tag{I.44}$$

$$u_{xx} + \nu u_y = 0 \quad \text{on } F, \tag{I.45}$$

$$\partial u/\partial n = f \quad \text{on } S, \tag{I.46}$$

$$u_y = 0 \quad \text{when } y = -d, \tag{I.47}$$

where $\nu = g/U^2$ in (I.45), and $f = U\mathbf{n} \cdot \mathbf{x}$ in (I.46) (by $\mathbf{n}$ and $\mathbf{x}$ we denote