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Abelian Varieties, Theta Functions and the Fourier Transform

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1

Line Bundles on Complex Tori

In this chapter we study holomorphic line bundles on complex tori, i.e., quotients of complex vector spaces by integral lattices. The main result is an explicit description of the group of isomorphism classes of holomorphic line bundles on a complex torus T. The topological type of a complex line bundle L on T is determined by its first Chern class $c_1(L) \in H^2(T, \mathbb{Z})$. This cohomology class can be interpreted as a skew-symmetric bilinear form $E : \Gamma \times \Gamma \to \mathbb{Z}$, where $\Gamma = H_1(T, \mathbb{Z})$ is the lattice corresponding to T. The existence of a holomorphic structure on L is equivalent to the compatibility of E with the complex structure on $\Gamma \otimes \mathbb{R}$ by which we mean the identity $E(iv, iv') = E(v, v')$. On the other hand, the group of isomorphism classes of topologically trivial holomorphic line bundles on T can be easily identified with the dual torus $T^\vee = \mathrm{Hom}(\Gamma, U(1))$. Now the set of isomorphism classes of holomorphic line bundles on T with the fixed first Chern class E is a $T^\vee$-torsor[1]. It can be identified with the $T^\vee$-torsor of quadratic maps $\alpha : \Gamma \to U(1)$ whose associated bilinear map $\Gamma \times \Gamma \to U(1)$ is equal to $\exp(\pi i E)$. These results provide a crucial link between the theory of theta functions and geometry that will play an important role throughout the first part of this book.

The holomorphic line bundle on T corresponding to a skew-symmetric form E and a quadratic map α as above, is constructed explicitly by equipping the trivial line bundle over a complex vector space with an action of an integral lattice. We show that as a result, every holomorphic line bundle on T has a canonical Hermitian metric and a Hermitian connection. We also show that the dual torus, $T^\vee$, has a natural complex structure and the universal family $\mathcal{P}$ of line bundles on T parametrized by $T^\vee$ (called the *Poincaré bundle*) has a natural holomorphic structure that we describe. In Chapter 9 we will study a purely algebraic version of this duality for abelian varieties.

[1] Following Grothendieck, we will use the term *G-torsor* when referring to a principal homogeneous space for a group G.

1.1. Cohomology of the Structure Sheaf

Let V be a finite-dimensional complex vector space, Γ be a lattice in V (i.e., Γ is a finitely generated $\mathbb{Z}$-submodule of V such that the natural map $\mathbb{R} \otimes_{\mathbb{Z}} \Gamma \to V$ is an isomorphism).

Definition. The complex manifold $T = V/\Gamma$ is called a *complex torus.*

As a topological space T is just a product of circles, so the cohomology ring $H^*(T, \mathbb{Z}) = \oplus_r H^r(T, \mathbb{Z})$ (resp., $H^*(T, \mathbb{R})$) can be identified naturally with the exterior algebra $\bigwedge^* H^1(T, \mathbb{Z})$ (resp., $\bigwedge^* H^1(T, \mathbb{R})$). Furthermore, we have a natural isomorphism $\Gamma \xrightarrow{\sim} H_1(T, \mathbb{Z})$ sending $\gamma \in \Gamma$ to the cycle $\mathbb{R}/\mathbb{Z} \to T : t \mapsto t\gamma$. Therefore, we get canonical isomorphisms $H^*(T, \mathbb{Z}) \simeq \bigwedge^* \Gamma^\vee$ and $H^*(T, \mathbb{R}) \simeq \bigwedge^* \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$, where $\Gamma^\vee = \mathrm{Hom}(\Gamma, \mathbb{Z})$ is the lattice dual to Γ.

Recall that one has the direct sum decomposition

$$V \otimes_{\mathbb{R}} \mathbb{C} = V \oplus \overline{V},$$

where V is identified with the subset of $V \otimes_{\mathbb{R}} \mathbb{C}$ consisting of vectors of the form $v \otimes 1 - iv \otimes i$, $\overline{V}$ is the complex conjugate subspace consisting of vectors $v \otimes 1 + iv \otimes i$. We also have the corresponding decomposition

$$\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) = V^\vee \oplus \overline{V}^\vee,$$

where $V^\vee = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ is the dual complex vector space to V, $\overline{V}^\vee$ is the space of $\mathbb{C}$-antilinear functionals on V. Since T is a Lie group, the tangent bundle to T is trivial and the above decomposition is compatible with the decomposition of the bundle of complex valued 1-forms on T according to types (1, 0) and (0, 1). Hence, we have canonical isomorphisms

$$\mathcal{E}^{p,q} \simeq \bigwedge^p V^\vee \otimes_{\mathbb{C}} \bigwedge^q \overline{V}^\vee \otimes_{\mathbb{C}} \mathcal{E}^{0,0},$$

where $\mathcal{E}^{p,q}$ is the sheaf of smooth (p, q)-forms on T.

The first basic result about T as a complex manifold is the calculation of cohomology of the structure sheaf $\mathcal{O}$, i.e., the sheaf of holomorphic functions.

Proposition 1.1. *One has a canonical isomorphism* $H^r(T, \mathcal{O}) \simeq \bigwedge^r \overline{V}^\vee$.

Proof. To calculate cohomology of $\mathcal{O}$ one can use the Dolbeault resolution:

$$0 \to \mathcal{O} \to \mathcal{E}^{0,0} \xrightarrow{\overline{\partial}} \mathcal{E}^{0,1} \xrightarrow{\overline{\partial}} \mathcal{E}^{0,2} \to \cdots.$$

We can consider elements of $\bigwedge^p \overline{V}^\vee$ as translation-invariant $(0, p)$-forms on T. Note that translation-invariant forms are automatically closed. We claim that this gives an embedding

$$i : \bigwedge^p \overline{V}^\vee \hookrightarrow H^p(T, \mathcal{O}).$$

Indeed, let $\int : \mathcal{E}^{0,0} \to \mathbb{C}$ be the integration map (with respect to some translation-invariant volume form on T) normalized by the condition $\int 1 = 1$. Then we can extend $\int$ to the map $\int : \mathcal{E}^{0,p} \to \bigwedge^p \overline{V}^\vee$. It is easy to see that $\int \circ \overline{\partial} = 0$, so $\int$ induces the map on cohomology

$$\int : H^p(T, \mathcal{O}) \to \bigwedge^p \overline{V}^\vee$$

such that $\int \circ i = \mathrm{id}$. Hence, i is an embedding. Let Ω^q be the sheaf of holomorphic q-forms on T. Since $\Omega^q \simeq \bigwedge^q V^\vee \otimes \mathcal{O}$, there is an induced embedding

$$i : \oplus_{p,q} \bigwedge^q V^\vee \otimes \bigwedge^p \overline{V}^\vee \to \oplus_{p,q} H^p(T, \Omega^q).$$

Notice that the source of this embedding can be identified with $H^*(T, \mathbb{C}) \simeq \bigwedge^*(V^\vee \oplus \overline{V}^\vee)$. Recall that for every Kähler complex compact manifold X one has Hodge decomposition $H^*(X, \mathbb{C}) \simeq \oplus_{p,q} H^p(X, \Omega^q)$ (e.g., [52], Chapter 0, Section 7). Since any translation-invariant Hermitian metric on T is Kähler, it follows that $\dim H^*(T, \mathbb{C}) = \dim \oplus_{p,q} H^p(T, \Omega^q)$. Therefore, the embedding i is an isomorphism. □

1.2. Appell–Humbert Theorem

It is well known that all holomorphic line bundles on $\mathbb{C}^n$ are trivial. Indeed, from the exponential exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \overset{\exp}{\to} \mathcal{O}^* \to 0 \tag{1.2.1}$$

we see that it suffices to prove triviality of $H^1(\mathbb{C}^n, \mathcal{O})$. But $H^{>0}(\mathbb{C}^n, \mathcal{O}) = 0$ by Poincaré $\overline{\partial}$-lemma ([52], Chapter 0, Section 2.)

For every complex manifold X we denote by $\mathrm{Pic}(X)$ the Picard group of X, i.e., the group of isomorphism classes of holomorphic line bundles on X. Triviality of $\mathrm{Pic}(\mathbb{C}^n)$ leads to the following computation of $\mathrm{Pic}(T)$ in terms of group cohomology of the lattice Γ.

Proposition 1.2. *Every holomorphic line bundle L on T is a quotient of the trivial bundle over V by the action of Γ of the form $\gamma(z, v) = (e_\gamma(v)z, v + \gamma)$,* *where* $\gamma \in \Gamma$, $z \in \mathbb{C}$, $v \in V$, *for some* 1*-cocycle* $\gamma \mapsto e_\gamma$ *of* Γ *with values in the group* $\mathcal{O}^*(V)$ *of invertible holomorphic functions on* V. *Here the action of* Γ *on* $\mathcal{O}^*(V)$ *is induced by its action on* V. *This correspondence extends to an isomorphism of groups*

$$\mathrm{Pic}(T) \simeq H^1(\Gamma, \mathcal{O}^*(V)).$$

Proof. Let $\pi : V \to T$ be the canonical projection. Since $\mathrm{Pic}(V)$ is trivial, for every holomorphic line bundle L on T the line bundle π^*L on V is trivial. Choose a trivialization $\pi^*L \simeq \mathcal{O}_V$. Then the natural action of Γ on π^*L becomes an action on the trivial bundle, which should be of the form stated in formulation for some collection $(e_\gamma(v), \gamma \in \Gamma)$ of invertible holomorphic functions on V. Unravelling the definition of the action we get the following condition on these functions:

$$e_{\gamma+\gamma'}(v) = e_\gamma(v+\gamma')e_{\gamma'}(v)$$

for every $\gamma, \gamma' \in \Gamma$. This is precisely the cocycle equation for the map $\Gamma \to \mathcal{O}^*(V) : \gamma \mapsto e_\gamma$. If we change the trivialization by another one, the function $e_\gamma(v)$ gets replaced by $e_\gamma(v)f(v+\gamma)f(v)^{-1}$ where f is an invertible holomorphic function on V. In other words, the cocycle $\gamma \mapsto e_\gamma$ changes by a coboundary. Thus, we get an isomorphism of $\mathrm{Pic}(T)$ with $H^1(\Gamma, \mathcal{O}^*(V))$. □

Definition. We will call 1-cocycles $\Gamma \to \mathcal{O}^*(V) : \gamma \mapsto e_\gamma$ *multiplicators* defining a line bundle on T.

From the exponential sequence (1.2.1) we get the long exact sequence

$$\begin{aligned} 0 \to H^1(T,\mathbb{Z}) \to H^1(T,\mathcal{O}) \to H^1(T,\mathcal{O}^*) \\ \xrightarrow{\delta} H^2(T,\mathbb{Z}) \to H^2(T,\mathcal{O}) \to \cdots. \end{aligned}$$

Recall that the first Chern class $c_1(L) \in H^2(T,\mathbb{Z})$ of a line bundle L on T is defined as the image of the isomorphism class $[L] \in H^1(T,\mathcal{O}^*)$ under the boundary homomorphism δ. We can consider $c_1(L)$ as a skew-symmetric bilinear form $\Gamma \times \Gamma \to \mathbb{Z}$. Note that $c_1(L)$ determines L as a topological (or C^∞) line bundle. Indeed, this follows immediately from the exponential sequence for continuous (resp., C^∞) functions and from the fact that the sheaf of continuous (resp., C^∞) functions is flabby.

The following natural problems arise.

1. Find out which topological line bundles admit a holomorphic structure, that is, describe the image of δ.

2. For every topological type of holomorphic line bundles find convenient multiplicators producing it.
3. Describe the group of topologically trivial holomorphic line bundles on T.

The solution of these problems is given in Theorem 1.3. The main ingredient of the answer is the following construction of multiplicators. Let H be a Hermitian form[2] on V, $E = \operatorname{Im} H$ be the corresponding skew-symmetric $\mathbb{R}$-bilinear form on V. Assume that E takes integer values on $\Gamma \times \Gamma$. Let $\alpha : \Gamma \to U(1) = \{z \in \mathbb{C} : |z| = 1\}$ be a map such that

$$\alpha(\gamma_1 + \gamma_2) = \exp(\pi i E(\gamma_1, \gamma_2))\alpha(\gamma_1)\alpha(\gamma_2) \tag{1.2.2}$$

(such α always exists; see Exercise 7). Set

$$e_\gamma(v) = \alpha(\gamma)\exp\left(\pi H(v, \gamma) + \frac{\pi}{2} H(\gamma, \gamma)\right),$$

where $\gamma \in \Gamma$, $v \in V$. It is easy to check that $\gamma \mapsto e_\gamma$ is a 1-cocycle. We denote by $L(H, \alpha)$ the corresponding holomorphic line bundle on T.

It is easy to see that

$$L(H_1, \alpha_1) \otimes L(H_2, \alpha_2) \simeq L(H_1 + H_2, \alpha_1\alpha_2),$$
$$[-1]^* L(H, \alpha) = L(H, \alpha^{-1}),$$

where $[-1] : T \to T$ is the involution of T sending v to $-v$.

Definition. Let E be a skew-symmetric $\mathbb{R}$-bilinear form on V. We say that E is *compatible with the complex structure* if $E(iv, iw) = E(v, w)$. We will say that E is *strictly compatible* with the complex structure if in addition $E(iv, v) > 0$ for $v \neq 0$.

Remark. In some books the definition of compatibility of E with the complex structure is equivalent to the strict compatibility in our definition. Note that strict compatibility implies that E is nondegenerate.

Theorem 1.3.

(1) A skew-symmetric bilinear form $E : \Gamma \times \Gamma \to \mathbb{Z}$ is the first Chern class of some holomorphic line bundle on T if and only if E (extended to an $\mathbb{R}$-bilinear form on V) is compatible with the complex structure on V.

[2] By a Hermitian form we mean an $\mathbb{R}$-bilinear form, which is $\mathbb{C}$-linear in the first argument and $\mathbb{C}$-antilinear in the second argument (no positivity condition is imposed).

(2) A skew-symmetric $\mathbb{R}$-bilinear form E on V is compatible with complex structure if and only if there exists a Hermitian form H on V such that $E = \operatorname{Im} H$ (then such H is unique). Assume in addition that E takes integer values on $\Gamma \times \Gamma$. Then there exists a map $\alpha : \Gamma \to U(1)$ satisfying (1.2.2), and for every such α one has $c_1(L(H, \alpha)) = -E$.

(3) The map $\alpha \mapsto L(0, \alpha)$ defines an isomorphism from $\operatorname{Hom}(\Gamma, U(1))$ to the group of isomorphism classes of topologically trivial holomorphic line bundles on T.

Proof. 1. Consider the canonical map

$$H^r(T, \mathbb{C}) \to H^r(T, \mathcal{O})$$

We can identify $H^r(T, \mathbb{C})$ with $\bigwedge^r (V \otimes_{\mathbb{R}} \mathbb{C})^\vee$. We have a decomposition $V \otimes_{\mathbb{R}} \mathbb{C} \simeq V \oplus \overline{V}$, and it is easy to see that the above map is given by restricting an alternating r-form from $V \otimes_{\mathbb{R}} \mathbb{C}$ to $\overline{V}$. Now consider the composed map

$$H^2(T, \mathbb{R}) \to H^2(T, \mathbb{C}) \to H^2(T, \mathcal{O}).$$

An element in $H^2(T, \mathbb{R})$ corresponds to a skew-symmetric real bilinear form E on V. The above map sends it to a $\mathbb{C}$-bilinear form on $\overline{V}$ obtained by extending scalars to $\mathbb{C}$ and restricting the form to the subspace $\overline{V} \subset V \otimes_{\mathbb{R}} \mathbb{C}$. The latter subspace consists of elements of the form $v \otimes 1 + iv \otimes i \in V \otimes_{\mathbb{R}} \mathbb{C}$. Thus, the condition that E maps to zero in $H^2(T, \mathcal{O})$ means that

$$(E \otimes \mathbb{C})(v \otimes 1 + iv \otimes i, w \otimes 1 + iw \otimes i) = 0$$

for any $v, w \in V$. It is easy to see that this condition is equivalent to $E(iv, iw) = E(v, w)$. Thus, a skew-symmetric bilinear form $E : \Gamma \times \Gamma \to \mathbb{Z}$ comes from a holomorphic line bundle if and only if it is compatible with a complex structure.

2. The Hermitian form H is constructed from E by the formula $H(v, w) = E(iv, w) + iE(v, w)$. It is easy to see that in this way we get a bijective correspondence between Hermitian forms and skew-symmetric $\mathbb{R}$-bilinear forms compatible with complex structure.

Now assume that E takes integer values on $\Gamma \times \Gamma$. The proof of existence of a map α satisfying (1.2.2) is sketched in Exercise 7. It remains to check that the class $c_1(L(H, \alpha)) \in H^2(T, \mathbb{Z})$ corresponds to the skew-symmetric form $-E$. By general nonsense (see Exercise 5) the coboundary map $H^1(T, \mathcal{O}^*) \to H^2(T, \mathbb{Z})$ can be identified with the coboundary map

$$\delta : H^1(\Gamma, \mathcal{O}^*(V)) \to H^2(\Gamma, \mathbb{Z}) \simeq \bigwedge^2 \Gamma.$$

The value of the latter map on a 1-cocycle $\gamma \mapsto e_\gamma(v)$ can be computed as follows. For every $\gamma \in \Gamma$ choose a holomorphic function f_γ on V such that $e_\gamma(v) = \exp(2\pi i f_\gamma(v))$. Then the 2-cocycle

$$F(\gamma_1, \gamma_2) = t_{\gamma_1}^* f_{\gamma_2} - f_{\gamma_1+\gamma_2} + f_{\gamma_1}$$

takes values in $\mathbb{Z}$ and represents $\delta(e_\gamma)$. Under the natural isomorphism $H^2(\Gamma, \mathbb{Z}) \simeq \mathrm{Hom}(\bigwedge^2 \Gamma, \mathbb{Z})$ the class of the 2-cocycle $F(\gamma_1, \gamma_2)$ corresponds to the skew-symmetric form

$$F(\gamma_2, \gamma_1) - F(\gamma_1, \gamma_2)$$

(see Exercise 6). It follows that the first Chern class of the line bundle associated with a 1-cocycle $\gamma \mapsto e_\gamma$ is represented by the skew-symmetric form

$$f_{\gamma_2}(v + \gamma_1) - f_{\gamma_1}(v + \gamma_2) + f_{\gamma_1}(v) - f_{\gamma_2}(v).$$

In our case we can take

$$f_\gamma(v) = \delta(\gamma) + \frac{1}{2i} H(v, \gamma) + \frac{1}{4i} H(\gamma, \gamma),$$

where $\alpha(\gamma) = \exp(2\pi i \delta(\gamma))$, which implies that $c_1(L(H, \alpha))$ corresponds to the form $-E$.

3. Consider the following exact sequence

$$0 \to H^1(T, \mathbb{Z}) \to H^1(T, \mathbb{R}) \to H^1(T, U(1)) \to H^2(T, \mathbb{Z}) \to H^2(T, \mathbb{R}).$$

The last arrow is injective, therefore, the map $H^1(T, \mathbb{R}) \to H^1(T, U(1))$ is surjective. On the other hand, the map $H^1(T, \mathbb{R}) \to H^1(T, \mathcal{O})$ is an isomorphism, so from the commutative diagram

$$\begin{array}{ccccc} H^1(\mathbb{Z}) & \longrightarrow & H^1(T, \mathbb{R}) & \longrightarrow & H^1(T, U(1)) \\ \downarrow & & \downarrow & & \downarrow \\ H^1(\mathbb{Z}) & \longrightarrow & H^1(T, \mathcal{O}) & \longrightarrow & H^1(T, \mathcal{O}^*) \end{array} \qquad (1.2.3)$$

we deduce that the image of $H^1(T, \mathcal{O}) \to H^1(T, \mathcal{O}^*)$ coincides with the image of the injective map $H^1(T, U(1)) \to H^1(T, \mathcal{O}^*)$. Note that we have a natural isomorphism $H^1(T, U(1)) \simeq \mathrm{Hom}(\Gamma, U(1))$. It is easy to check that in terms of this isomorphism the embedding $H^1(T, U(1)) \to H^1(T, \mathcal{O}^*) = \mathrm{Pic}(T)$ is given by $\alpha \mapsto L(0, \alpha)$. $\square$

As a corollary, we get the following description of $\mathrm{Pic}(T)$ due to Appell and Humbert.

Corollary 1.4. *The group* $\mathrm{Pic}(T)$ *is isomorphic to the group of pairs* (H, α)*, where* H *is a Hermitian form on* V *such that* $E = \mathrm{Im}\, H$ *takes integer values on* Γ*,* α *is a map from* Γ *to* $U(1)$ *such that (1.2.2) is satisfied. The group law on the set of pairs is given by* $(H_1, \alpha_1)(H_2, \alpha_2) = (H_1 + H_2, \alpha_1\alpha_2)$.

The only nonobvious part of the above argument is the invention of line bundles $L(H, \alpha)$. We will see in Section 2.5 that in the case of positive definite H their construction is quite natural from the point of view of the Heisenberg group.

1.3. Metrics and Connections

The line bundle $L(H, \alpha)$ constructed in Section 1.2 comes equipped with a natural Hermitian metric. To construct it, first we define a metric on the trivial line bundle on V by setting

$$h(v) = \exp(-\pi H(v, v)).$$

Proposition 1.5. *The metric* h *descends to a metric on* $L(H, \alpha)$*. There is a unique connection on* $L(H, \alpha)$ *that is compatible with this metric and with the complex structure on* $L(H, \alpha)$*. Its curvature is equal to* $\pi i E$ *considered as a translation-invariant* 2*-form on* T*, where* $E = \mathrm{Im}\, H$.

Proof. It is easy to check that the metric h is invariant with respect to the action of Γ on the trivial bundle, which we used to define $L(H, \alpha)$. Therefore, it descends to a metric on $L(H, \alpha)$. It is well known that for every Hermitian metric on a holomorphic line bundle there exists a unique connection compatible with this metric and the complex structure ([52], Chapter 0, Section 5). To describe this connection on $L(H, \alpha)$ we are going to write the corresponding Γ-invariant connection ∇ on the trivial line bundle on V. The section

$$s = \exp\left(\frac{\pi}{2} H(v, v)\right)$$

of the trivial bundle on V is orthonormal with respect to our metric h. Hence, we should have

$$\langle \nabla s, s\rangle + \langle s, \nabla s\rangle = 0. \tag{1.3.1}$$

We can write $\nabla s = ds + s\omega$ for some $(1, 0)$-form ω, where

$$ds = \frac{\pi}{2}(H(dv, v) + H(v, dv))s.$$

Here the notation $H(dv, v)$ and $H(v, dv)$ should be understood as follows. Let us identify V with $\mathbb{C}^n$ in such a way that $H(z, z') = \sum_{i=1}^r z_i \overline{z_i'}$, where r is the rank of H. Then we have $H(dv, v) = \sum_{i=1}^r \overline{z_i} dz_i$, etc. Now we can rewrite equation (1.3.1) as

$$\pi(H(dv, v) + H(v, dv)) + \omega + \overline{\omega} = 0.$$

This implies that $\omega = -\pi H(dv, v)$. Thus, we obtain

$$\nabla = d - \pi H(dv, v).$$

The curvature of this connection is equal to $\pi H(dv, dv)$. If we identify V with $\mathbb{C}^n$ as above then $H(dv, dv) = \sum_{i=1}^r dz_i \wedge d\overline{z_i}$. Note that $\overline{H(dv, dv)} = -H(dv, dv)$. Hence, the curvature is equal to

$$\pi H(dv, dv) = \frac{\pi}{2}(H - \overline{H})(dv, dv) = \pi i E(dv, dv). \qquad \square$$

In the case $H = 0$ we obtain that the line bundle $L(0, \alpha)$, where $\alpha \in \mathrm{Hom}(\Gamma, U(1))$, can be equipped with a flat unitary connection compatible with the complex structure. It is not difficult to check that the corresponding 1-dimensional representation of the fundamental group $\pi_1(T) = \Gamma$ is given by the character α.

1.4. Poincaré Line Bundle

According to Theorem 1.3, topologically trivial holomorphic line bundles on T are parametrized (up to an isomorphism) by the group $T^\vee = \mathrm{Hom}(\Gamma, U(1))$. Note that we have the following isomorphisms:

$$\mathrm{Hom}(\Gamma, U(1)) = \mathrm{Hom}(\Gamma, \mathbb{R})/\,\mathrm{Hom}(\Gamma, \mathbb{Z}) = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})/\,\Gamma^\vee.$$

Also, one has a canonical isomorphism $\overline{V}^\vee \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ sending a $\mathbb{C}$-antilinear map $\phi : V \to \mathbb{C}$ to $\mathrm{Im}\,\phi$. Hence, we can identify $T^\vee = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})/\,\Gamma^\vee$ with the complex torus $\overline{V}^\vee/\,\Gamma^\vee$. It is easy to see that $\Gamma^\vee \subset \overline{V}^\vee$ coincides with the set of all $\mathbb{C}$-antilinear maps $\phi : V \to \mathbb{C}$ such that $\mathrm{Im}\,\phi(\Gamma) \subset \mathbb{Z}$.

Definition. $T^\vee$ is called the *dual complex torus to T*.

Since $T^\vee$ parametrizes all topologically trivial line bundles on T, it is natural to expect that there is a universal line bundle on $T \times T^\vee$. Such a line bundle is constructed in the following definition.

Definition. The *Poincaré line bundle* is the holomorphic line bundle $\mathcal{P}$ on $T \times T^\vee = (V \oplus \overline{V}^\vee)/(\Gamma \oplus \Gamma^\vee)$ obtained as $L(H_{\text{univ}}, \alpha_{\text{univ}})$, where H_{univ} is the natural Hermitian form on $V \oplus \overline{V}^\vee$:

$$H_{\text{univ}}((v, \phi), (v', \phi')) = \phi(v') + \overline{\phi'(v)},$$
$$\alpha_{\text{univ}}(\gamma, \gamma^\vee) = \exp(\pi i \langle \gamma^\vee, \gamma \rangle).$$

For every $\alpha \in T^\vee$ we have a natural isomorphism of holomorphic bundles on T

$$\mathcal{P}|_{T \times \{\alpha\}} \simeq L(0, \alpha).$$

Furthermore, every (holomorphic) family of topologically trivial line bundles on T parametrized by a complex manifold S is induced by $\mathcal{P}$ via a holomorphic map $S \to T^\vee$.

In Part II we will consider an algebraic analogue of duality between complex tori. The corresponding algebraic Poincaré line bundle will be the main ingredient in the definition of the Fourier–Mukai transform in Chapter 11.

Exercises

1. Let $f : V \to V'$ be a $\mathbb{C}$-linear map of complex vector spaces mapping a lattice $\Gamma \subset V$ into a lattice $\Gamma' \subset V'$. Then f induces the holomorphic map $f : T = V/\Gamma \to T' = V'/\Gamma'$ of the corresponding complex tori. Show that for a line bundle $L(H, \alpha)$ on T' associated with a Hermitian form H on V' and a map $\alpha : \Gamma' \to U(1)$ as in Section 1.2 one has

$$f^* L(H, \alpha) \simeq L(f^* H, f^* \alpha).$$

2. Let $t_v : V \to V : x \mapsto x + v$ be a translation. Prove that

$$t_v^* L(H, \alpha) \simeq L(H, \alpha \cdot \nu_v),$$

where $\nu_v(\gamma) = \exp(2\pi i E(v, \gamma))$. Check that this isomorphism is compatible with metrics introduced in Section 1.3 up to a constant factor.

3. Let $E : \Gamma \times \Gamma \to \mathbb{Z}$ be a skew-symmetric form compatible with the complex structure on V. Let $\Gamma_0 \subset \Gamma$ be the kernel of E. Prove that $V_0 = \mathbb{R}\Gamma_0$ is a complex subspace of V.
4. Let $L(H, \alpha)$ be a holomorphic line bundle on T corresponding to some data (H, α) as in Section 1.2. Let $V_0 \subset V$ be the kernel of $E = \operatorname{Im} H$, $\Gamma_0 = V_0 \cap \Gamma$. Assume that $\alpha|_{\Gamma_0} \equiv 1$. Prove that L is a pull-back of a holomorphic line bundle on $T' = V/V_0 + \Gamma$ under the natural projection $T \to T'$.
5. In this exercise *a sheaf* always means a sheaf of abelian groups. A Γ-equivariant sheaf on V is a sheaf $\mathcal{F}$ on V equipped with the system of isomorphisms $i_\gamma : t_\gamma^*\mathcal{F} \simeq \mathcal{F}$, where $t_\gamma : V \to V$ is the translation by γ. These isomorphisms should satisfy the following cocycle condition:

$$i_{\gamma+\gamma'} = i_\gamma \circ t_\gamma^*(i_{\gamma'}).$$

We denote by $\Gamma - \mathrm{Sh}_V$ the category of Γ-equivariant sheaves on V and by Sh_T the category of sheaves on T.

(a) Show that the functor $\pi^{\cdot}$ establishes an equivalence of categories $\mathrm{Sh}_T \xrightarrow{\sim} \Gamma - \mathrm{Sh}_V$. Deduce that if $\mathcal{F}$ is an injective sheaf on T then $\pi^{\cdot}\mathcal{F}$ is an injective object in the category $\Gamma - \mathrm{Sh}_V$.

(b) Let $\mathcal{F}$ be a sheaf on T. Construct a functorial isomorphism $H^0(T, \mathcal{F}) \to H^0(V, \pi^{\cdot}\mathcal{F})^\Gamma$.

(c) Let $\mathbb{Z}_V$ denotes the constant sheaf on V corresponding to $\mathbb{Z}$. Then for every Γ-module M the constant sheaf $M \otimes \mathbb{Z}_V$ on V is equipped with a natural Γ-action. Show that for every Γ-equivariant sheaf $\mathcal{G}$ on V there is a functorial isomorphism

$$\mathrm{Hom}_\Gamma(M, H^0(V, \mathcal{G})) \simeq \mathrm{Hom}_{\Gamma-\mathrm{Sh}_V}(M \otimes \mathbb{Z}_V, \mathcal{G}).$$

Deduce from this that if $\mathcal{G}$ is an injective object of the category $\Gamma - \mathrm{Sh}_V$ then $H^0(V, \mathcal{G})$ is an injective Γ-module.

(d) Let $\mathcal{F}$ be an injective sheaf on T. Show that $H^{>0}(V, \pi^{\cdot}\mathcal{F}) = 0$. [*Hint*: Use the fact that π is a local homeomorphism to show that $\pi^{\cdot}\mathcal{F}$ is flabby.]

(e) Let $\mathcal{F}$ be a sheaf on T such that $H^{>0}(V, \pi^{\cdot}\mathcal{F}) = 0$. Choose an injective resolution $\mathcal{F}_\bullet$ of $\mathcal{F}$. Prove that cohomology of the complex $H^0(V, \pi^{\cdot}\mathcal{F}_\bullet)^\Gamma$ can be identified with $H^*(\Gamma, H^0(V, \pi^{\cdot}\mathcal{F}))$. Now using (b) construct isomorphisms

$$H^i(T, \mathcal{F}) \to H^i(\Gamma, H^0(V, \pi^{\cdot}\mathcal{F})).$$

Show that if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of sheaves such that $\pi^{\cdot}\mathcal{F}'$, $\pi^{\cdot}\mathcal{F}$ and $\pi^{\cdot}\mathcal{F}''$ are acyclic, then the above maps fit into a morphism of long exact sequences.

(f) Show that the sheaf-theoretic pull-back of the exponential exact sequence on T gives the exponential exact sequence on V.

(g) Prove that the global exponential sequence on V

$$0 \to \mathbb{Z} \to \mathcal{O}(V) \to \mathcal{O}^*(V) \to 0$$

is exact.

(h) Identify the connecting homomorphism $H^1(T, \mathcal{O}^*) \to H^2(T, \mathbb{Z})$ with the connecting homomorphism in group cohomology $H^1(\Gamma, \mathcal{O}^*(V)) \to H^2(\Gamma, \mathbb{Z})$.

6. The goal of this exercise is to identify the isomorphism $i : H^2(\Gamma, \mathbb{R}) \simeq H^2(T, \mathbb{R})$ obtained in the previous exercise with the natural map $H^2(\Gamma, \mathbb{R}) \to \mathrm{Hom}(\bigwedge^2 \Gamma, \mathbb{R})$ sending a 2-cocycle $c : \Gamma \times \Gamma \to \mathbb{R}$ to the skew-symmetric bilinear form $c(\gamma_2, \gamma_1) - c(\gamma_1, \gamma_2)$.

(a) Show that the (real) de Rham complex on V: $\mathcal{E}^0(V) \to \mathcal{E}^1(V) \to \cdots$ is a resolution of $\mathbb{R}$ by acyclic Γ-modules. Derive from this the following description of the isomorphism i. Start with a 2-cocycle $c : \Gamma \times \Gamma \to \mathbb{R}$ of Γ with coefficients in $\mathbb{R}$. Then there exists a collection of smooth functions f_γ on V such that

$$c(\gamma_1, \gamma_2) = t_{\gamma_1}^* f_{\gamma_2} - f_{\gamma_1+\gamma_2} + f_{\gamma_1},$$

where $c(\gamma_1, \gamma_2)$ is considered as a constant function on V. Next, there exists a 1-form ω on V such that $df_\gamma = t_\gamma^*\omega - \omega$ for every γ. This implies that the 2-form $d\omega$ is Γ-invariant. Hence, it descends to a closed 2-form on T. Its cohomology class is $i(c)$.

(b) Recall that the isomorphism $H^2(T, \mathbb{R}) \to \mathrm{Hom}(\bigwedge^2 \Gamma, \mathbb{R}) = \mathrm{Hom}_{\mathbb{R}}(\bigwedge_{\mathbb{R}}^2 V, \mathbb{R})$ sends the cohomology class of a closed 2-form η on T to $\int \eta$, where the map

$$\int : \mathcal{E}^2(T) \to \mathrm{Hom}_{\mathbb{R}}\left(\bigwedge\nolimits_{\mathbb{R}}^2 V, \mathbb{R}\right)$$

is obtained from the isomorphism

$$\mathcal{E}^2(T) \simeq \mathrm{Hom}_{\mathbb{R}}\left(\bigwedge\nolimits_{\mathbb{R}}^2 V, \mathbb{R}\right) \otimes \mathcal{E}^0(T)$$

via the integration map $\int : \mathcal{E}^0(T) \to \mathbb{R}$. Choosing real coordinates on V associated with a basis of Γ show that this map sends the 2-form on T representing $i(c)$ to the skew-symmetric bilinear form $c(\gamma_2, \gamma_1) - c(\gamma_1, \gamma_2)$.

7. Let $\overline{E} : \Gamma \times \Gamma \to \mathbb{Z}/2\mathbb{Z}$ be a skew-symmetric bilinear form modulo 2 (*skew-symmetry* means that $\overline{E}(\gamma, \gamma) = 0$ for every $\gamma \in \Gamma$). Prove that there exists a map $f : \Gamma \to \mathbb{Z}/2\mathbb{Z}$, such that

$$\overline{E}(\gamma_1, \gamma_2) = f(\gamma_1 + \gamma_2) + f(\gamma_1) + f(\gamma_2).$$

Deduce that for every skew-symmetric bilinear form $E : \bigwedge^2 \Gamma \to \mathbb{Z}$ there exists a map $\alpha : \Gamma \to \{\pm 1\}$ satisfying (1.2.2).

8. Let T be a complex torus, $e_1, \dots, e_{2n}$ be the basis of the lattice $H^1(T, \mathbb{Z})$, $e_1^*, \dots, e_{2n}^*$ be the dual basis of $H^1(T^\vee, \mathbb{Z})$, where $T^\vee$ is the dual torus. Show that the first Chern class of the Poincaré bundle on $T \times T^\vee$ is given by

$$c_1(\mathcal{P}) = \sum_{i=1}^{2n} e_i \wedge e_i^*.$$