

# Floer homology groups in Yang–Mills theory

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with the assistance of M. Furuta and D. Kotschick



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# 1

## Introduction

In 1985 Andreas Floer discovered new topological invariants of certain 3-manifolds, the ‘Floer homology groups’. This book originated from a series of seminars on this subject held in Oxford in 1988, the manuscript for the book being written sporadically over the intervening 12 years. The original plan of the project has been modified over time, but the basic aims have remained largely the same: these are, first, to give a thorough exposition of Floer’s original work, and, second, to develop some further aspects of the theory which have not appeared in detail in the literature before. The author can only apologise for the long delay in completing this project.

Floer’s original motivation for introducing his groups – beyond the intrinsic interest and beauty of the construction – seems to have been largely as a source of new invariants in 3-manifold theory, refining the Casson invariant which had been discovered shortly before. It was soon realised however that Floer’s conception fitted in perfectly with the ‘instanton invariants’ of 4-dimensional manifolds, which date from much the same period. Roughly speaking, the Floer groups are the data required to extend this theory from closed 4-manifolds to manifolds with boundary. From another point of view the Floer groups appear, formally, as the homology groups in the ‘middle dimension’ of an infinite-dimensional space (the space of connections modulo equivalence) associated to a 3-manifold. This picture is obtained by carrying certain aspects of the Morse theory description of the homology of a finite-dimensional manifold over to infinite dimensions. All of this is closely related to ideas from quantum field theory – indeed, one of Floer’s starting points was the renowned paper of Witten, [49], which *inter alia* forged a link between quantum mechanics and Morse theory

– and the connection with mathematical physics permeates the whole subject.

The formal properties of the Floer groups, and their relation with invariants in four dimensions, fit into a general conceptual framework of ‘topological quantum field theories’ which was propounded in the late 1980s by Segal, Atiyah, Witten and others. We recall from [2] that a topological field theory, in  $d + 1$  dimensions, consists of two functors on manifolds. The first assigns to each closed, oriented,  $d$ -manifold  $Y$  a vector space  $\mathcal{H}(Y)$  (over, say, the complex numbers). The second assigns to each compact, oriented  $(d + 1)$ -dimensional manifold  $X$  with boundary  $Y$  a vector

$$Z(X) \in \mathcal{H}(Y).$$

These are required to satisfy three axioms:

- (1) The vector space assigned to a disjoint union  $Y_1 \cup Y_2$  is the tensor product

$$\mathcal{H}(Y_1 \cup Y_2) = \mathcal{H}(Y_1) \otimes \mathcal{H}(Y_2).$$

- (2)  $\mathcal{H}(\bar{Y}) = \mathcal{H}(Y)^*$ , where  $\bar{Y}$  is  $Y$  with the reversed orientation.  
 (3) Suppose  $X$  is a  $(d + 1)$ -manifold with boundary (which may be disconnected), and that  $X$  contains  $Y$  and  $\bar{Y}$  as two of its boundary components. Let  $X^\sharp$  be the oriented manifold obtained from  $X$  by identifying these two boundary components. Then we require that

$$Z(X^\sharp) = c(Z(X)),$$

where the contraction  $c : \mathcal{H}(\partial X) \rightarrow \mathcal{H}(\partial X^\sharp)$  is induced from the dual pairing  $\mathcal{H}(Y) \otimes \mathcal{H}(\bar{Y}) \rightarrow \mathbf{C}$  and the decomposition

$$\mathcal{H}(\partial X) = \mathcal{H}(Y) \otimes \mathcal{H}(\bar{Y}) \otimes \mathcal{H}(\partial X^\sharp).$$

These axioms have some simple consequences. First, Axiom 1 implies that if  $Y = \emptyset$  is the empty  $d$ -manifold then  $\mathcal{H}(\emptyset)$  is canonically isomorphic to  $\mathbf{C}$ . Thus if  $X$  is a *closed*  $(d + 1)$ -manifold the vector  $Z(X)$  is a numerical invariant of  $X$ . Second, suppose that a  $(d + 1)$ -manifold  $U$  is a cobordism from  $Y_1$  to  $Y_2$ , so the oriented boundary of  $U$  is a disjoint union  $\bar{Y}_1 \cup Y_2$ . Then, by Axioms 1 and 2,  $Z(U)$  is an element of  $\mathcal{H}(Y_1)^* \otimes \mathcal{H}(Y_2)$  and hence gives a linear map

$$\zeta_U : \mathcal{H}(Y_1) \rightarrow \mathcal{H}(Y_2).$$

If  $V$  is a cobordism from  $Y_2$  to a third manifold  $Y_3$  then Axiom 3 states that

$$\zeta_{V \circ U} = \zeta_V \circ \zeta_U : \mathcal{H}(Y_1) \rightarrow \mathcal{H}(Y_3),$$

where  $V \circ U$  is the obvious composite cobordism. So we obtain a functor from the category of  $d$ -manifolds, with morphisms defined by cobordisms, to the category of vector spaces and linear maps.

The original motivation which led Segal and others to develop this kind of axiomatic picture was to abstract in a tidy mathematical form the basic structure of quantum field theories (more precisely, of conformal field theory on Riemann surfaces). The theories which are usually considered in physics differ from the set-up considered above in that they operate on manifolds with some additional differential-geometric structure, for example a Riemannian metric or a conformal structure. It is precisely the absence of these geometric structures in our set-up which leads to the designation *topological* quantum field theories, and which means that we obtain topological (or, more precisely, differential-topological) invariants of manifolds. In a typical physical set-up the corresponding space  $\mathcal{H}(Y)$  would be an infinite-dimensional Hilbert space defined, at least schematically, by associating to  $Y$  a space of ‘fields’  $\mathcal{C}(Y)$  (an element of  $\mathcal{C}(Y)$  might be a tensor field over  $Y$ ), and then letting  $\mathcal{H}(Y)$  be a space of  $L^2$  functions on  $\mathcal{C}(Y)$ . The vector  $Z(X)$  is obtained by functional integration over a space of fields on  $X$ , with given boundary value on  $Y$ .

The Yang–Mills invariants, and Floer groups, fit into this general scheme, with  $d = 3$ . In outline, for a 3-manifold  $Y$ , we take the Floer groups (with complex co-efficients say)

$$\mathcal{H}(Y) = HF_*(Y).$$

For a closed 4-manifold  $X$  the Yang–Mills instantons define a numerical invariant  $Z(X)$ , and for a 4-manifold with boundary we obtain invariants with values in the Floer homology of the boundary. Actually, as we shall see, the simple axioms above need to be modified slightly to apply to the Yang–Mills set-up and the theory has a number of special features. For example, the invariants of a closed 4-manifold are not in general just numbers but functions on the homology of the manifold – so we might regard the functor as being defined on a category of 4-manifolds containing preferred homology classes. Nevertheless these axioms capture the essence of the matter. Contrasting with the physical set-up outlined above, we can say very roughly that in place of



the infinite-dimensional space  $\mathcal{C}(Y)$  of *all* fields (i.e. connections) we restrict in this topological theory to the *finite* set of flat connections (modulo equivalence) over  $Y$ , and we restrict to ‘instanton’ connections over 4-manifolds, so that in place of the functional integration over connections we now have merely to *count* the instantons with given flat boundary values. To make rigorous sense of this, a key step is to add half-infinite tubes to our 4-manifolds, so that we have a picture in which the boundary is ‘at infinity’.

An important goal then of this book is to develop this picture, of the Floer groups as part of a topological field theory, in detail. It is important to emphasise at the outset that, even after all this time, we are not able to complete this task. On the one hand there are, as we shall see, rather fundamental technical reasons why one cannot expect to have this simple picture without imposing some restrictions on the manifolds which are considered. On the other hand, even within the confines of the theory that one might reasonably hope for, there are crucial technical difficulties, arising from the non-compactness of instanton moduli spaces which – despite much labour by many mathematicians – have not yet been fully overcome. Failing, therefore, a definitive treatment we round off the book, in Chapter 8, by seeking to explain the problems that remain, and further developments one may expect in the future. We shall see that – far from being dull, technical matters – these difficulties lead to striking and unexpected formulae involving classical special functions.

Throughout the early 1990s an important motivation for the development of Floer theory was the hope that this might lead to new calculations of 4-manifold invariants, via cutting and pasting techniques. It has to be said that, at least on a narrow interpretation, this programme did not yield as much fruit as one might have hoped, and its goals have been to a large extent overtaken by events. The main lines of progress in this area (aside from algebro-geometrical techniques) came roughly thus. Firstly, through work of Mrowka and others involving cutting and pasting along 3-tori which, while it probably could be incorporated in a suitable generalisation of Floer theory, was not formulated explicitly in these terms. Secondly, through work of Kronheimer and Mrowka using singular connections (although again a version of Floer theory appeared in their arguments). Thirdly, and most decisively, through the introduction in 1994 of the Seiberg–Witten invariants. Leaving aside the well-known issue of the equivalence of the two theories, this last gives a more economical and powerful basis for the entire subject and makes

the older instanton theory largely redundant as far as applications to 4-manifold topology go.

While it cannot be denied that the material in this book is less topical now than a decade ago (and at some points the text may have a slightly dated air, reflecting the long period over which it has been written) the author hopes that it is still worthwhile to present this material. We mention three grounds for this hope. First, the main thrust of the first part of the book is to develop certain differential-geometric and analytical techniques which apply to a wide range of problems, going beyond Yang–Mills theory (for example to the analogous symplectic Floer theory, to the Seiberg–Witten version of the Floer theory, to gluing problems for other structures such as self-dual metrics and metrics of special holonomy). Second, Floer’s fundamental idea of defining ‘middle-dimensional homology’ for suitable infinite-dimensional manifolds is such an appealing one, and again one which in principle could appear in many different contexts, that it seems to deserve a thorough treatment. Third, while, as we have said above, some of the original motivation for the theory *vis-à-vis* 4-manifold topology is now reduced, there are intriguing questions which remain to be settled in setting up the Floer theory and understanding the whole relation between the instanton invariants and the Seiberg–Witten invariants. Some of these, in particular the appearance of modular forms, are touched on in Chapter 8. The Seiberg–Witten version of the Floer theory is a topic which is being very actively developed at the time of writing and, in conjunction with Floer’s original groups, is expected to have important consequences in 3-manifold theory.

There are many topics omitted from this book. (In some cases these are things which we had hoped to include, in earlier and more ambitious plans, but found the energy wanting when it came to the point.) There are absolutely no examples: this is an entirely ‘theoretical’ treatment. We do not discuss the Casson invariant of homology spheres [46], or Floer’s exact surgery sequence [8]. We do not mention Fukaya’s extension of Floer’s homology groups [9]. We do not have anything to say explicitly about the related theories developed by Taubes [47] and Morgan, Mrowka and Rubermann [36]. We do not say anything about the various interesting links between Floer’s theory and the moduli spaces of flat connections over surfaces, and with algebraic geometry. Finally we say nothing about many of the deeper and more recent developments, connected with the Seiberg–Witten theory, such as the work of Muñoz [37] and Froyshov [25] on the ‘finite type’ condition. We

do not discuss the Seiberg–Witten equations [51], [13], and the variant of the Floer theory they define. Except for the discussion of Fintushel and Stern’s work of 1993 in Chapter 8 we have confined ourselves to an exposition of ideas that were current *circa* 1990.

On the other hand, we do digress from the narrow goal of setting up the Floer theory at a number of points. Thus, for example, we develop some of the main analytical results (in Chapter 4) in more generality than we need, because the ideas seem interesting and useful in other applications. We attempt to say a little about the background in mathematical physics, and the analogy with the symplectic theory.

The general scheme of the book is as follows. The first part (Chapters 2–5) aims to give a complete definition of the Floer groups of a homology 3-sphere: essentially following Floer’s original paper. Chapter 6 develops the basic connection with 4-manifold invariants. The thrust of the first part is towards the geometrical and analytical techniques: at the beginning of Chapter 6 we step back to discuss the overall conceptual picture. Some readers may wish to look at the beginning of Chapter 6 at an earlier stage. Chapter 7 is devoted to refinements of the theory, mainly involving ideas from algebraic topology. This sets the stage for Chapter 8 in which, as we have mentioned, we discuss open problems and likely further developments.

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