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# Collocation Methods for Volterra Integral and Related Functional Differential Equations

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# Contents

# Preface

The principal aims of this monograph are (i) to serve as an introduction and a guide to the basic principles and the analysis of collocation methods for a broad range of functional equations, including initial-value problems for ordinary and delay differential equations, and Volterra integral and integro-differential equations; (ii) to describe the current 'state of the art' of the field; (iii) to make the reader aware of the many (often very challenging) problems that remain open and which represent a rich source for future research; and (iv) to show, by means of the annotated list of references and the Notes at the end of each chapter, that Volterra equations are not simply an 'isolated' small class of functional equations but that they play an (increasingly) important – and often unexpected! – role in time-dependent PDEs, boundary integral equations, and in many other areas of analysis and applications.

The book can be divided in a natural way into four parts:

- In Part I we focus on collocation methods, mostly in piecewise polynomial spaces, for first-kind and second-kind Volterra integral equations (VIEs, *Chapter 2*), and Volterra integro-differential equations (*Chapter 3*) possessing *smooth solutions*: here, the regularity of the solution on the interval of integration essentially coincides with that of the given data. This situation is similar to the one encountered in initial-value problems for ordinary differential equations. Hence, *Chapter 1* serves as an introduction to collocation methods applied to initial-value problems for ODEs: this will allow us to acquire an appreciation of the richness of these methods and their analysis for more general functional equations encountered in subsequent chapters of this book.
- Part II deals with Volterra integral and integro-differential equations containing *delay arguments*. For non-vanishing delays (*Chapter 4*), smooth data will in general no longer lead to solutions with comparable regularity on the entire

interval of integration, and hence optimal orders of convergence in collocation approximations comparable to those seen in the previous chapters can only be attained by a careful choice of the underlying meshes. For equations with (vanishing) proportional delays (*Chapter 5*) the situation is completely different. Here, the solution inherits the regularity of the given data, but on uniform meshes the analysis of the attainable order of superconvergence is much more complex, due to the 'overlap' between the collocation points and their images under the given delay function. This is not yet completely understood, and a number of problems remain open.

- In Part III we study collocation methods for Volterra integral equations (*Chapter 6*) and integro-differential equations (*Chapter 7*) with *weakly singular kernels*. The presence of these kernel singularities gives rise to a singular behaviour (different in nature from the non-smooth behaviour encountered in Chapter 4) of the solutions at the initial point of the interval of integration, and at the primary discontinuity points if there is a non-vanishing delay: typically, the first- or second-order derivatives of the solutions, or (in the case of first-kind Volterra integral equations) the solution itself, are unbounded at these points. Thus, a decrease in the order of convergence can only be avoided either by introducing suitably graded meshes, or by switching to appropriate non-polynomial spline spaces, reflecting the nature of this singular behaviour. This insight is then combined with results gained in Chapter 4 when turning, at the end of Chapters 6 and 7, to collocation methods for Volterra equations possessing weakly singular kernels *and* delay arguments.
- In Part IV (*Chapters 8 and 9*) we shall have reached the current 'frontier' in the analysis of collocation methods when considering their use for solving integral-algebraic equations (IAEs, which may be viewed as differential-algebraic equations (DAEs) with memory terms, or as 'abstract' DAEs in an infinite-dimensional setting) and singularly perturbed Volterra integral and integro-differential equations. It is known from the numerical analysis of DAEs that the 'direct' application of collocation (even for index-1 problems) will in general not yield the 'expected' convergence (and stability) behaviour since very often the given problem is not 'numerically well formulated'. But while this is now well understood for DAEs, we have a far way to go when analysing collocation methods for suitably reformulated IAEs. Thus, much of Chapter 8 consists of a look into the future. Chapter 9 adds some additional dimensions to this outlook: it points to a number of – to me – promising and important directions of research that may contain the keys to obtaining deeper insight into a number of the open problems we met in previous chapters.

It will become apparent that the number of unanswered questions and open problems becomes larger as we move through the chapters. For example, the analysis of asymptotic stability of collocation solutions for most classes of Volterra integral and functional differential equations is still in its infancy (I believe that relatively little essential progress has been made since Pieter van der Houwen and I wrote down a similar observation in the preface of our 1986 book), and this lack of progress and new results is reflected in the fact that the present monograph deals with this topic only peripherally. It has also become clear from recent advances in the analysis of the asymptotic properties of numerical solutions to ordinary differential equations (Hairer and Wanner (1996)), dynamical systems (Stuart and Humphries (1996)), and delay differential equations (Bellen and Zennaro (2003)), that the study of the analogous properties of collocation methods for more general functional differential and integral equations will eventually have to be treated in a separate monograph.

Most chapters begin with a section reviewing the relevant elementary theory of the class of equations to be discretised by collocation. It goes without saying that a thorough understanding of the theoretical aspects of a given functional equation is imperative since a successful analysis of its discretisation will often be inspired, and thus helped along, by insight into the essential features in the analysis of the given equation and the corresponding discrete analogue derived by collocation.

At the end of each chapter the reader will find exercises and extensive notes. The *Exercises* range from 'hands-on' problems (intended to illustrate and complement the theory of the respective chapter) to research topics of various degree of difficulty, and these will often include important unsolved problems. The purpose of the *Notes* is twofold: they contain remarks complementing the contents of the given chapter (giving, e.g., the sources of original results), and they point out papers on related topics not treated in the book.

The list of *References* tries to be representative, without being exhaustive, of the developments in the research on collocation methods over the last 80 years or so. Moreover, it includes many papers on the analysis and application of collocation methods to types of functional equations not treated in this book. The intent of these references is to guide the reader to work that describes results and mathematical techniques whose analogues and application are, in my view, of potential interest for Volterra integral and related functional differential equations, and they may thus yield the motivation for future research work. In order to make this extensive bibliography more useful and give it a certain guiding role, many of its items have been annotated, so as to enhance the Notes given at the end of each chapter: the brief comments are either cross-references to related work, give an idea of the main content of a paper, or point to books and survey articles containing large bibliographies complementing the one given in this monograph.

As mentioned above, the bibliography lists also many papers and books dealing with topics where exciting work is currently being carried but which, due to limitations of space (and lack of expertise on my part) are not included in this book. Among these topics are *spectral and pseudo-spectral methods* (which appear to be very promising for Volterra equations but whose theory remains to be developed); *sequential (collocation based) regularisation methods* for first-kind VIEs; the numerical treatment of *Volterra equations occurring in control theory*; and *a posteriori error estimation* and the design of *adaptive collocation methods* (especially for problems with non-smooth solutions). I hope that these additional references, while not directly relevant to the text of the monograph, and the accompanying notes will encourage the reader to have a closer look at these important topics.

This monograph is intended for researchers in numerical and applied analysis, for 'users' of collocation methods in the physical sciences and in engineering, and as an introduction to collocation methods for senior undergraduate and graduate students.

Since the exercise section of each chapter contains a rich list of *open problems*, the book may also serve as a source of topics for M.Sc. and Ph.D. theses.

*Prerequisites:* Senior-level courses in linear algebra, the theory of ordinary differential equations, and numerical analysis (especially numerical quadrature and the numerical solution of ODEs). A knowledge of elementary functional analysis will prove helpful in Chapter 8.

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