# 1

# The collocation method for ODEs: an introduction

A collocation solution $u_h$ to a functional equation (for example an ordinary differential equation or a Volterra integral equation) on an interval $I$ is an element from some finite-dimensional function space (the collocation space) which satisfies the equation on an appropriate finite subset of points in $I$ (the set of collocation points) whose cardinality essentially matches the dimension of the collocation space. If initial (or boundary) conditions are present then $u_h$ will usually be required to fulfil these conditions, too.

The use of polynomial or piecewise polynomial collocation spaces for the approximate solution of boundary-value problems has its origin in the 1930s. For initial-value problems in ordinary differential equations such collocation methods were first studied systematically in the late 1960s: it was then shown that collocation in continuous piecewise polynomial spaces leads to an important class of implicit (high-order) Runge–Kutta methods.

## 1.1 Piecewise polynomial collocation for ODEs

### 1.1.1 Collocation-based implicit Runge–Kutta methods

Consider the initial-value problem

$$y'(t) = f(t, y(t)), \quad t \in I := [0, T], \quad y(0) = y_0, \tag{1.1.1}$$

and assume that the (Lipschitz-) continuous function $f : I \times \Omega \subset \mathbb{R} \to \mathbb{R}$ is such that (1.1.1) possesses a unique solution $y \in C^1(I)$ for all $y_0 \in \Omega$. Let

$$I_h := \{t_n : 0 = t_0 < t_1 < \ldots < t_N = T\}$$

be a given (not necessarily uniform) mesh on $I$, and set $\sigma_n := (t_n, t_{n+1}]$, $\bar{\sigma}_n := [t_n, t_{n+1}]$, with $h_n := t_{n+1} - t_n$ $(n = 0, 1, \ldots, N-1)$. The quantity

$h := \max\{h_n : 0 \le n \le N-1\}$ will be called the *diameter* of the mesh $I_h$; in the context of time-stepping we will also refer to $h$ as the *stepsize*. Note that we have, in rigorous notation,

$$t_n = t_n^{(N)}, \quad \sigma_n := \sigma_n^{(N)}, \quad h_n = h_n^{(N)} \ (n = 0, 1, \ldots, N-1), \quad \text{and} \ \ h = h^{(N)}.$$

However, we will usually suppress this dependence on $N$, the number of subintervals corresponding to a given mesh $I_h$, except occasionally in the convergence analyses where $N \to \infty$, $h = h^{(N)} \to 0$, so that $Nh^{(N)}$ remains uniformly bounded.

The solution $y$ of the initial-value problem (1.1.1) will be approximated by an element $u_h$ of the piecewise polynomial space

$$S_m^{(0)}(I_h) := \{v \in C(I) : \ v|_{\bar{\sigma}_n} \in \pi_m \ (0 \le n \le N-1)\}, \tag{1.1.2}$$

where $\pi_m$ denotes the space of all (real) polynomials of degree not exceeding $m$. It is readily verified that $S_m^{(0)}(I_h)$ is a linear space whose dimension is

$$\dim S_m^{(0)}(I_h) = Nm + 1$$

(a description of more general piecewise polynomial spaces will be given in Section 2.2.1). This approximation $u_h$ will be found by *collocation*; that is, by requiring that $u_h$ satisfy the given differential equation on a given suitable finite subset $X_h$ of $I$, and coincide with the exact solution $y$ at the initial point $t = 0$. It is clear that the cardinality of $X_h$, the *set of collocation points*, will have to be equal to $Nm$, and the obvious choice of $X_h$ is to place $m$ distinct collocation points in each of the $N$ subintervals $\bar{\sigma}_n$. To be more precise, let $X_h$ be given by

$$X_h := \{t = t_n + c_i h_n : \ 0 \le c_1 < \ldots < c_m \le 1 \ (0 \le n \le N-1)\}. \tag{1.1.3}$$

For a given mesh $I_h$, the *collocation parameters* $\{c_i\}$ completely determine $X_h$. Its cardinality is

$$|X_h| = \begin{cases} Nm & \text{if } \ 0 < c_1 < \ldots < c_m \le 1 \ (\text{or } \ 0 \le c_1 < \ldots < c_m < 1), \\ N(m-1)+1 & \text{if } \ 0 = c_1 < c_2 < \ldots < c_m = 1 \ \ (m \ge 2). \end{cases}$$

The collocation solution $u_h \in S_m^{(0)}(I_h)$ for (1.1.1) is defined by the *collocation equation*

$$u_h'(t) = f(t, u_h(t)), \quad t \in X_h, \quad u_h(0) = y(0) = y_0. \tag{1.1.4}$$

If $u_h$ corresponds to a set of collocation points with $c_1 = 0$ and $c_m = 1$ $(m \ge 2)$, it lies (if it exists on $I$) in the smoother space $S_m^{(0)}(I_h) \cap C^1(I) =: S_m^{(1)}(I_h)$ of dimension $N(m-1)+2$ whenever the given function $f$ in (1.1.1) is continuous. This follows readily by considering the collocation equation (1.1.4) at

$t = t_{n-1} + c_m h_{n-1} =: t_n^-$ and at $t = t_n + c_1 h_n =: t_n^+$: taking the difference and using the continuity of $f$ leads to

$$u_h'(t_n^+) - u_h'(t_n^-) = 0, \quad n = 1, \ldots, N-1,$$

and this is equivalent to $u_h'$ being continuous at $t = t_n$.

In order to obtain more insight into this piecewise polynomial collocation method, and to exhibit its recursive nature, we now derive the computational form of (1.1.4). This will reveal that the collocation equation (1.1.4) represents the stage equations of an *m-stage continuous implicit Runge–Kutta method* for the initial-value problem (1.1.1) (compare also the original papers by Guillou and Soulé (1969), Wright (1970), or the book by Hairer, Nørsett and Wanner (1993).

Here, and in subsequent chapters of the book, it will be convenient (and natural) to work with the local Lagrange basis representations of $u_h$. Since $u_h'|_{\sigma_n} \in \pi_{m-1}$, we have

$$u_h'(t_n + vh_n) = \sum_{j=1}^{m} L_j(v) Y_{n,j}, \quad v \in (0, 1], \quad Y_{n,j} := u_h'(t_n + c_j h_n), \tag{1.1.5}$$

where the polynomials

$$L_j(v) := \prod_{k \neq j}^{m} \frac{v - c_k}{c_j - c_k} \qquad (j = 1, \ldots, m),$$

denote the Lagrange fundamental polynomials with respect to the (distinct) collocation parameters $\{c_i\}$. Setting $y_n := u_h(t_n)$ and

$$\beta_j(v) := \int_0^v L_j(s)ds \quad (j = 1, \ldots, m),$$

we obtain from (1.1.5) the local representation of $u_h \in S_m^{(0)}(I_h)$ on $\bar{\sigma}_n$, namely

$$u_h(t_n + vh_n) = y_n + h_n \sum_{j=1}^{m} \beta_j(v) Y_{n,j}, \quad v \in [0, 1]. \tag{1.1.6}$$

The unknown (derivative) approximations $Y_{n,i}$ $(i = 1, \ldots, m)$ in (1.1.6) are defined by the solution of a system of (generally nonlinear) algebraic equations obtained by setting $t = t_{n,i} := t_n + c_i h_n$ in the collocation equation (1.1.4) and employing the local representations (1.1.5) and (1.1.6). This system is

$$Y_{n,i} = f\left(t_{n,i}, y_n + h_n \sum_{j=1}^{m} a_{i,j} Y_{n,j}\right), \quad (i = 1, \ldots, m), \tag{1.1.7}$$

where we have defined $a_{i,j} := \beta_j(c_i)$.

We see that the equations (1.1.6) and (1.1.7) define, as asserted above, a *continuous implicit Runge–Kutta (CIRK) method* for the initial-value problem (1.1.1): its $m$ stage values $Y_{n,i}$ are given by the solution of the nonlinear algebraic systems (1.1.7), and (1.1.6) defines the approximation $u_h$ for each $t \in \bar{\sigma}_n$ $(n = 0, 1, \ldots, N-1)$. This local representation may be viewed as the natural interpolant in $\pi_m$ on $\bar{\sigma}_n$ for the data $\{(t_n, y_n),\ (t_{n,i}, Y_{n,i})\ (i = 1, \ldots, m)\}$.

It thus follows that such a continuous implicit RK method contains an embedded 'classical' (discrete) $m$-stage implicit Runge–Kutta method for (1.1.1): it corresponds to (1.1.6) with $v = 1$,

$$y_{n+1} := u_h(t_n + h_n) = y_n + h_n \sum_{j=1}^{m} b_j Y_{n,j} \quad (n = 0, 1, \ldots, N-1), \tag{1.1.8}$$

with $b_j := \beta_j(1)$, and the stage equations (1.1.7).

If $m \geq 2$ and if the collocation parameters $\{c_i\}$ are such that

$$0 = c_1 < c_2 < \ldots < c_m = 1,$$

then $t_{n,1} = t_n$ implies $Y_{n,1} = f(t_n, y_n)$, and the CIRK method (1.1.6), (1.1.7) reduces to

$$u_h(t_n + vh_n) = y_n + h_n \beta_1(v) f(t_n, y_n) + h_n \sum_{j=2}^{m} \beta_j(v) Y_{n,j}, \quad v \in [0, 1], \tag{1.1.9}$$

and

$$Y_{n,i} = f\left(t_{n,i}, y_n + h_n a_{i,1} f(t_n, y_n) + h_n \sum_{j=2}^{m} a_{i,j} Y_{n,j}\right) \quad (i = 2, \ldots, m). \tag{1.1.10}$$

Moreover, since $c_m = 1$, we obtain

$$Y_{n,m} = f\left(t_{n+1}, y_n + h_n b_1 f(t_n, y_n) + h_n \sum_{j=2}^{m} b_j Y_{n,j}\right).$$

**Example 1.1.1** $u_h \in S_1^{(0)}(I_h)$ $(m = 1)$, with $c_1 =: \theta \in [0, 1]$:
Since $L_1(v) \equiv 1$ and $\beta_1(v) = v$ (hence $a_{1,1} = \theta$ and $b_1 = 1$), (1.1.6) reduces to

$$u_h(t_n + vh_n) = y_n + h_n v Y_{n,1}, \quad v \in [0, 1],$$

with $Y_{n,1}$ defined by the solution of

$$Y_{n,1} = f(t_n + \theta h_n, y_n + h_n \theta Y_{n,1}).$$

These equations may be combined into a single one (by setting $v = 1$ in the expression for $u_h(t_n + vh_n)$ and solving for $Y_{n,1}$); the resulting method is the

*continuous $\theta$-method* for (1.1.1),

$$u_h(t_n + vh_n) = (1 - v)y_n + vy_{n+1}, \quad v \in [0, 1].$$

where

$$y_{n+1} = y_n + h_n f(t_n + \theta h_n, (1 - \theta)y_n + \theta y_{n+1})$$

implicitly defines $y_{n+1}$.

This family of continuous one-stage Runge–Kutta methods contains the *continuous implicit Euler method* ($\theta = 1$) and the *continuous implicit midpoint method* ($\theta = 1/2$). For $\theta = 0$ we obtain the *continuous explicit Euler method*. Due to its importance in the time-stepping of (spatially) semidiscretised parabolic PDEs (or PVIDEs) we state the continuous implicit midpoint method for the *linear* ODE

$$y'(t) = a(t)y(t) + g(t), \quad t \in I,$$

with $a$ and $g$ in $C(I)$. Setting $\theta = 1/2$ we obtain

$$y_{n+1} = y_n + \frac{h_n}{2} a(t_n + h_n/2)[y_n + y_{n+1}] + g(t_n + h_n/2)(n = 0, 1, \ldots, N-1),$$

or, using the notation $t_{n+1/2} := t_n + h_n/2$,

$$\left(1 - \frac{h_n}{2} a(t_{n+1/2})\right) y_{n+1} = \left(1 + \frac{h_n}{2} a(t_{n+1/2})\right) y_n + h_n g(t_{n+1/2}). \quad (1.1.11)$$

Observe the difference between (1.1.11) and the *continuous trapezoidal method*: the latter corresponds to collocation in the space $S_2^{(0)}(I_h)$, with $c_1 = 0,\ c_2 = 1$ being the Lobatto points; it is described in Example 1.1.2 below ($m = 2$).

**Example 1.1.2** $u_h \in S_2^{(0)}(I_h)$ ($m = 2$), with $0 \le c_1 < c_2 \le 1$:
It follows from $L_1(v) = (c_2 - v)/(c_2 - c_1),\ L_2(v) = (v - c_1)/(c_2 - c_1)$ that

$$\beta_1(v) = \frac{v(2c_2 - v)}{2(c_2 - c_1)}, \qquad \beta_2(v) = \frac{v(v - 2c_1)}{2(c_2 - c_1)}.$$

Hence, $b_1 = \beta_1(1) = (2c_2 - 1)/(2(c_2 - c_1)),\ b_2 = \beta_2(1) = (1 - 2c_1)/(2(c_2 - c_1))$. The resulting continuous two-stage Runge–Kutta method thus reads:

$$u_h(t_n + vh_n) = y_n + h_n\{\beta_1(v)Y_{n,1} + \beta_2(v)Y_{n,2}\}, \quad v \in [0, 1],$$

where

$$Y_{n,i} = f(t_{n,i}, y_n + h_n\{a_{i,1}Y_{n,1} + a_{i,2}Y_{n,2}\}) \quad (i = 1, 2).$$

We present three important special cases:

- *Gauss points* $c_1 = (3-\sqrt{3})/6,\ c_2 = (3+\sqrt{3})/6$:
  We obtain

$$\beta_1(v) = v(1+\sqrt{3}(1-v))/2, \qquad \beta_2(v) = v(1-\sqrt{3}(1-v))/2,$$

  and

$$A := [\, a_{i,j} \,] = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} - \frac{\sqrt{3}}{6} \\ \frac{1}{4} + \frac{\sqrt{3}}{6} & \frac{1}{4} \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

  The discrete version of this *two-stage implicit Runge–Kutta–Gauss method* (of order 4; cf. Section 1.1.3, Corollary 1.1.6) was introduced by Hammer and Hollingsworth (1955) and generalised by Kuntzmann in 1961 (see Ceschino and Kuntzmann (1963) for details).
- *Radau II points* $c_1 = 1/3,\ c_2 = 1$:
  Here, we have

$$\beta_1(v) = 3v(2-v)/4, \qquad \beta_2(v) = 3v(v-2/3)/4,$$

  and

$$A = \begin{bmatrix} \frac{5}{12} & -\frac{1}{12} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix}, \quad b = \begin{bmatrix} \frac{3}{4} \\ \frac{1}{4} \end{bmatrix}.$$

  This represents the continuous *two-stage Radau IIA method*.
- *Lobatto points* $c_1 = 0,\ c_2 = 1$ ($\Longrightarrow\ u_h \in S_2^{(1)}(I_h)$):
  The continuous weights are now

$$\beta_1(v) = v(2-v)/2, \qquad \beta_2(v) = v^2/2,$$

  and hence

$$A = \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad b = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

  This yields the *continuous trapezoidal method*: it can be written in the form

$$u_h(t_n + vh_n) = y_n + \frac{h_n}{2}\left(v(2-v)Y_{n,1} + v^2 Y_{n,2}\right), \quad v \in [0,1],$$

  with

$$Y_{n,1} = f(t_n, y_n), \quad Y_{n,2} = f(t_{n+1}, y_n + (h_n/2)\{Y_{n,1} + Y_{n,2}\}).$$

  (See also Hammer and Hollingsworth (1955).)

For the linear ODE $y'(t) = a(t)y(t) + g(t)$ the stage equation assumes the form

$$\left(1 - \frac{h_n a(t_{n+1})}{2}\right) Y_{n,2} = \left(1 + \frac{h_n a(t_n)}{2}\right) a(t_{n+1}) y_n + \frac{h_n a(t_{n+1})}{2} g(t_n) + g)t_{n+1}).$$

**Remark** Other examples of (discrete) RK methods based on collocation, including methods corresponding to the Radau I points ($c_1 = 0,\ c_2 = 2/3$ when $m = 2$), may be found for example in the books by Butcher (1987, 2003), Lambert (1991), and Hairer and Wanner (1996).

There is an alternative way to formulate the above continuous implicit Runge–Kutta method (1.1.6),(1.1.7). Setting

$$U_{n,i} := y_n + h_n \sum_{j=1}^{m} a_{i,j} Y_{n,j} \quad (i = 1, \ldots, m),$$

we obtain the *symmetric* formulation

$$u_h(t_n + vh_n) = y_n + h_n \sum_{j=1}^{m} \beta_j(v) f(t_{n,j}, U_{n,j}), \quad v \in [0, 1], \tag{1.1.12}$$

with

$$U_{n,i} = y_n + h_n \sum_{j=1}^{m} a_{i,j} f(t_{n,j}, U_{n,j}) \quad (i = 1, \ldots, m). \tag{1.1.13}$$

Here, the unknown stage values $U_{n,i}$ represent aproximations to the solution $y$ at the collocation points $t_{n,i}$ $(i = 1, \ldots, m)$. For $v = 1$, (1.1.12) yields the symmetric analogue of (1.1.8),

$$y_{n+1} = y_n + h_n \sum_{j=1}^{m} b_j f(t_{n,j}, U_{n,j}); \tag{1.1.14}$$

if $c_m = 1$ we have $y_{n+1} = U_{n,m}$.

For later reference, and to introduce notation needed later, we also write down the above CIRK method (1.1.6), (1.1.7) for the *linear* initial-value problem

$$y'(t) = a(t)y(t), \quad t \in I, \quad y(0) = y_0,$$

where $a \in C(I)$. Setting $A := (a_{i,j}) \in L(\mathbb{R}^m)$, $\beta(v) := (\beta_1(v), \ldots, \beta_m(v))^T \in \mathbb{R}^m$, and $\mathbf{Y}_n := (Y_{n,1}, \ldots, Y_{n,m})^T \in \mathbb{R}^m$, the CIRK method can be written in the form

$$u_h(t_n + vh_n) = y_n + h_n \beta^T(v) \mathbf{Y}_n, \quad v \in [0, 1], \tag{1.1.15}$$

with $\mathbf{Y}_n$ given by the solution of the linear algebraic system

$$[\mathcal{I}_m - h_n A_n]\mathbf{Y}_n = \mathrm{diag}(a(t_{n,i}))\mathbf{e} \cdot y_n \quad (n = 0, 1, \ldots, N-1). \tag{1.1.16}$$

Here, $\mathcal{I}_m$ denotes the identity in $L(\mathbb{R}^m)$, $A_n := \mathrm{diag}(a(t_{n,i}))A$, and $\mathbf{e} := (1, \ldots, 1)^T \in \mathbb{R}^m$.

The derivation of the analogue of (1.1.15),(1.1.16) corresponding to the symmetric formulation (1.1.12),(1.1.13) of the CIRK method is left as an exercise (Exercise 1.10.1).

The classical conditions for the existence and uniqueness of a solution $y \in C^1(I)$ to the initial-value problem (1.1.1) (see, e.g. Hairer, Nørsett and Wanner (1993, Sections I.7–I.9) assure the existence and uniqueness of the collocation solution $u_h \in S_m^{(0)}(I_h)$ to (1.1.1) or its linear counterpart for all $h := \max_{(n)} h_n$ in some interval $(0, \bar{h})$, provided that $f_y$ is bounded (or $a$ and $g$ lie in $C(I)$ when the ODE is $y' = a(t)y + g(t)$). In the latter case, the existence of such an $\bar{h}$ follows from the Neumann Lemma which states that $(\mathcal{I}_m - h_n A_n)^{-1}$ is uniformly bounded for all sufficiently small $h_n > 0$, so that $h_n ||A_n|| < 1$ for some (operator) matrix norm. We shall give the precise formulation of this result in in Chapter 3 (Theorem 3.2.1) for VIDEs which contains the version for ODEs as a special case.

It is clear that not every implicit Runge–Kutta method can be obtained by collocation as described above (see, for example, Nørsett (1980), Hairer, Nørsett and Wanner (1993)): a necessary condition is clearly that the parameters $c_i$ are distinct. The framework of *perturbed collocation* (Nørsett (1980), Nørsett and Wanner (1981); see also Section 1.2 below) encompasses all implicit) Runge–Kutta methods. There is also an elegant connection between continuous Runge–Kutta methods and *discontinuous collocation methods* (Hairer, Lubich and Wanner (2002, pp. 31–34)). The following result (which can be found in Hairer, Nørsett and Wanner (1993, p. 212)) characterises those implicit Runge–Kutta methods that are collocation-based.

**Theorem 1.1.1** *The $m$-stage implicit Runge–Kutta method defined by (1.1.7) and (1.1.8), with distinct parameters $c_i$ and order at least $m$, can be obtained by collocation in $S_m^{(0)}(I_h)$, as described above, if and only if the relations*

$$\sum_{j=1}^{m} a_{i,j} c_j^{\nu-1} = \frac{c_i^{\nu}}{\nu}, \quad \nu = 1, \ldots, m \quad (i = 1, \ldots, m),$$

*hold.*

The **proof** of this result is left as an exercise. Recall that a (discrete) Runge–Kutta method for (1.1.1) is said to be of *order $p$* if

$$|y(t_1) - y_1| \le Ch^p$$

for all sufficiently smooth $f = f(t, y)$ in (1.1.1). The next section will reveal that the collocation solution $u_h \in S_m^{(0)}(I_h)$ to (1.1.1) is of global order $p \geq m$ on $I$.

### 1.1.2 Convergence and global order on $I$

Suppose that the collocation equation (1.1.4) defines a unique collocation solution $u_h \in S_m^{(0)}(I_h)$ for all sufficiently small mesh diameters $h \in (0, \bar{h})$. What are the optimal values of $p_\nu$ and $p_\nu^*$ ($\nu = 0, 1$) in the (global and local) error estimates

$$||y^{(\nu)} - u_h^{(\nu)}||_\infty := \sup_{t \in I} |y^{(\nu)}(t) - u_h^{(\nu)}(t)| \leq C_\nu h^{p_\nu} \tag{1.1.17}$$

and

$$||y^{(\nu)} - u_h^{(\nu)}||_{h,\infty} := \max_{t \in I_h \setminus \{0\}} |y^{(\nu)}(t) - u_h^{(\nu)}(t)| \leq C_\nu h^{p_\nu^*}, \tag{1.1.18}$$

respectively? These values depend of course on the regularity of the solution $y$ of the initial-value problem (1.1.1). For arbitrarily regular $y$ we will refer to the largest attainable $p_\nu$ ($\nu = 0, 1$) as the *(optimal) orders of global (super-) convergence* (on the interval $I$) of $u_h$ and $u_h'$, respectively, and the corresponding $p_\nu^*$ will be called the *(optimal) orders of local superconvergence* (at the mesh points $I_h \setminus \{0\}$) of $u_h$ and $u_h'$, provided $p_\nu^* > p_\nu$.

In order to introduce the essential ideas underlying the answer to the above question regarding the optimal orders, we first present the result on global convergence for the *linear* initial-value problem

$$y'(t) = a(t)y(t) + g(t), \quad t \in I, \quad y(0) = y_0. \tag{1.1.19}$$

**Theorem 1.1.2** *Assume that*

*(a) the given functions in (1.1.19) satisfy $a, g \in C^m(I)$;*
*(b) the collocation solution $u_h \in S_m^{(0)}(I_h)$ for the initial-value problem (1.1.19) corresponding to the collocation points $X_h$ is defined by (1.1.15), (1.1.16);*
*(c) $\bar{h} > 0$ is such that, for any $h \in (0, \bar{h})$, each of the linear systems (1.1.16) has a unique solution.*

*Then the estimates*

$$||y - u_h||_\infty := \max_{t \in I} |y(t) - u_h(t)| \leq C_0 ||y^{(m+1)}||_\infty h^m \tag{1.1.20}$$

*and*

$$||y' - u_h'||_\infty := \sup_{t \in I} |y'(t) - u_h'(t)| \leq C_1 ||y^{(m+1)}||_\infty h^m, \tag{1.1.21}$$

*hold for $h \in (0, \bar{h})$ and any $X_h$ with $0 \le c_1 < \ldots < c_m \le 1$. The constants $C_\nu$ depend on the collocation parameters $\{c_i\}$ but are independent of $h$, and the exponent $m$ of $h$ cannot in general be replaced by $m+1$.*

**Proof** Assumption (a) implies that $y \in C^{m+1}(I)$ and hence $y' \in C^m(I)$. Thus we have, using Peano's Theorem (Corollary 1.8.2 with $d = m$) for $y'$ on $\bar{\sigma}_n$,

$$y'(t_n + vh_n) = \sum_{j=1}^{m} L_j(v) Z_{n,j} + h_n^m R_{m+1,n}^{(1)}(v), \quad v \in [0, 1], \tag{1.1.22}$$

with $Z_{n,j} := y'(t_{n,j})$. The Peano remainder term and Peano kernel are given by

$$R_{m+1,n}^{(1)}(v) := \int_0^1 K_m(v, z) y^{(m+1)}(t_n + zh_n) dz, \tag{1.1.23}$$

and

$$K_m(v, z) := \frac{1}{(m-1)!} \left\{ (v - z)_+^{m-1} - \sum_{k=1}^{m} L_k(v)(c_k - z)_+^{m-1} \right\}, \quad v \in [0, 1].$$

Integration of (1.1.22) leads to

$$y(t_n + vh_n) = y(t_n) + h_n \sum_{j=1}^{m} \beta_j(v) Z_{n,j} + h_n^{m+1} R_{m+1,n}(v), \quad v \in [0, 1], \tag{1.1.24}$$

where

$$R_{m+1,n}(v) := \int_0^v R_{m+1,n}^{(1)}(s) ds$$

(see also Exercise 1.10.3).

Recalling the local representation (1.1.6) of the collocation solution $u_h$ on $\bar{\sigma}_n$, and setting $\mathcal{E}_{n,j} := Z_{n,j} - Y_{n,j}$, the collocation error $e_h := y - u_h$ on $\bar{\sigma}_n$ may be written as

$$e_h(t_n + vh_n) = e_h(t_n) + h_n \sum_{j=1}^{m} \beta_j(v) \mathcal{E}_{n,j} + h_n^{m+1} R_{m+1,n}(v), \quad v \in [0, 1], \tag{1.1.25}$$

while

$$e_h'(t_n + vh_n) = \sum_{j=1}^{m} L_j(v) \mathcal{E}_{n,j} + h_n^m R_{m+1,n}^{(1)}(v), \quad v \in (0, 1], \tag{1.1.26}$$