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Introduction

The phases of QCD have presented a challenge to theoretical physics for many years. Originally it was considered folly to think about topics such as confinement, chiral-symmetry breaking, and different states of matter in relativistic field theories. Anything beyond traditional perturbation theory was met with skepticism. However, the development of semi-classical methods in field theory, the lattice-gauge formulation of the subject, and the generalizations of analytic methods of analysis from two-dimensional systems, such as duality and transitions driven by topological excitations, have changed all that.

This book is a look at the fundamentals of QCD from this perspective. After introducing lattice-gauge theory, beginning with fundamentals and reaching important recent developments, it will emphasize the application of QCD to the study of matter in extreme environments. Effective Lagrangians, which incorporate the constraints of low-energy dynamics and their symmetry realizations, will also be developed to provide a complementary, and insightful, perspective. Application of perturbative methods will also be presented in the regime of their validity.

Why extreme environments? A major theme of this book is the idea that, to understand the dynamics of QCD in ordinary circumstances, one needs to master them in extreme environments.

For example, to appreciate confinement, one can heat the system until the thermal fluctuations prevent the formation of the thin flux tubes which give rise to a linearly confining potential. In doing so, the special ingredients in the theory's dynamics which favor a vacuum pressure that squeezes flux into thin continuous tubes are emphasized. In addition, the essential ingredient of Gauss' law and exact color gauge symmetry will be seen to lie at the heart of the confining features of ordinary QCD.

Another example, which is central to this book, is afforded by chiral-symmetry breaking, the fact that the theory with massless quarks develops dynamical constituent quark masses consistent with its global chiral symmetries. Again, in ordinary environments quark–anti-quark attractive forces, generated by mechanisms such as perturbative gluon exchanges and nonperturbative instanton forces and even those of flux tubes and confinement, favor the appearance of a condensate of quark–anti-quark pairs in the system's vacuum. This condensate then gives the vacuum an indefinite chirality, which supports dynamically generated masses for its constituents as well as its colorless physical states. When such a vacuum is heated, the violent thermal fluctuations will melt the condensate and lead to restoration of chiral symmetry. The corresponding thermodynamic state, or thermal vacuum, however, is anything but simple. In this hot environment affecting the theory of massless quarks and gluons, we expect plasma formation and screening of color. The hot theory is qualitatively different from its cold relative: color screening replaces color confinement, massless quarks and gluons replace dynamical symmetry breaking, and a rich mass spectrum of colorless states with a large level spacing is replaced by a spectrum of screened states with a smaller level spacing.

Another extreme environment that lies on the experimental horizon is that of cold, dense matter. An example is afforded by the interior of a neutron star, where the baryon number density is far higher than that of ordinary nuclear matter. This environment is not well understood in theories like QCD, for reasons which will be discussed at length within this book. Since the discovery of asymptotic freedom in theories like QCD, it has been realized that matter should become weakly interacting if the Fermi energies of the quarks, controlled by the chemical potential μ, are large in comparison with the confinement scale, which is of the order of a few hundred MeV (or 1 fm^{-1}). It has only recently fully been appreciated that even this weakly coupled state is rich in intricate phenomena due to pairing instabilities similar to the phenomenon of superconductivity in metals. Reliable calculations are possible in this regime controlled by the smallness of the gauge coupling of strong interactions. However, this control is very quickly lost once the densities decrease, and weak-coupling approximations are not justified. From the lower-density side, we can reliably predict that, as the chemical potential of the baryon charge approaches the mass of the lightest baryon, M_{N} (i.e., the quark chemical potential approaches $M_{\mathrm{N}}/3$), a transition to a new state of matter occurs, since it becomes energetically favorable for the ground state to contain baryons. We also have empirical evidence leading us to believe that this state is the state of nuclear matter, chunks of which form nuclei of heavy elements, and that astronomically large chunks, held together by gravity, form neutron stars. But how does this matter, which we know is not very dense on the scale of QCD (one baryon per 6 fm^3) transform into asymptotically free quark matter, as the density, or μ, increases? What happens to the baryons, whose very existence seems doubtful in such dense *quark* matter? Is there a transition, and, if there is, at what μ and of what type? A hypothetical peek into the interior of a neutron star would reveal the answers. Such questions are still in the realm of speculation in QCD research at this time. One of the goals of this book is to provide the tools and background for a new generation of researchers to study this problem.

The challenge of QCD in cold, dense environments is a major theme of this book. We believe that there is a new field of science awaiting us there. It is the condensed matter or even the chemistry of QCD. The point is that, as cold QCD is subjected to a chemical potential, there will be the possibility of the creation of new states of extended QCD matter, analogous to the creation of new states of molecular matter in atomic physics. The phases of molecular matter are extraordinarily rich because molecules themselves have intricate three-dimensional symmetries and can interact among themselves via cooperative phenomena like those involving induced dipole–dipole forces, which generate energy scales several orders of magnitude smaller than the fundamental atomic energy scale, the Rydberg. These effects conspire to generate phase diagrams of molecules that are very complex and interesting, and determine the character of our environment. In the case of QCD, the quarks and gluons have internal symmetries described by their color and flavor indices. There are small symmetry breakings arising from the bare mass splittings between the quarks. There is a rich physical mass spectrum of colorless states with a level spacing of several hundred MeV. How will such a physical system respond to a chemical potential comparable to its level spacing? Such an environment will encourage the production of new, nontrivially ordered states of fundamental matter. Exotic phases consisting of meson condensates, superfluids, crystals, etc. have been proposed. From this point of view, it would be naive to expect reliable predictions here very soon, since such states of matter and their stability will depend on grasping the dynamics of QCD to within an accuracy of a few MeV. A goal, ambitious at that, for this book, is to encourage a new generation of physicists to tackle this new field of the extremely condensed matter, or chemistry, of QCD, which may very well determine our environment on the cosmic scale.

Why is so little known about QCD at nonzero μ and what are the present speculations about new states of matter at vanishing temperature and nonvanishing chemical potential? Lattice-gauge theory has made little progress on this subject because its action for the SU(3) color group in the presence of a nonzero baryon chemical potential becomes complex and not amenable to reliable methods of analysis, such as computer simulations. This problem will be dealt with at length in the text. We will review the random-matrix analysis of the failure of the quenched "approximation" of QCD to model it adequately at nonzero chemical potential. We will also see that models with positive fermion determinants can be analyzed by lattice and effective-Lagrangian methods. In fact the SU(2) color case can be studied in detail and new states of strongly interacting superfluids are predicted.

QCD of three colors, but with the chemical potential for the isospin charge I_3, is very similar to the SU(2) color case (and in fact also similar to the quenched approximation to QCD – as we shall see using the random-matrix approach). The pions condense into a pion superfluid at an easily predictable critical chemical potential.

We will also view the SU(3) theory in an environment where the chemical potential is asymptotically large and there is a huge Fermi sphere of weakly interacting colorful quarks. In this environment, which is amenable to the weak-coupling approach, one can identify a QCD analog of Bardeen–Cooper–Schriefer (BCS) fermion pairing and superconductivity – color superconductivity. This QCD phenomenon is much richer than its condensed-matter (QED) counterpart. It also has much more integrity in it, since in QED the dominant forces between electrons are repulsive and an intricate phonon-mediated interaction is needed in order to provide the necessary attraction. One needs more than just electrons and photons for traditional BCS fermion pairing. In QCD quarks and gluons and their mutual interactions are the only necessary ingredients. The attraction provided by gluons is long-ranged (on the typical scale of $1/\mu$ in such quark matter), thus providing a stronger effect than one could expect from a short-ranged attraction.

The phenomenon of color superconductivity is also richer than its QED counterpart due to there being a larger set of global symmetries, the chiral symmetries of QCD. An especially interesting theoretical case is the limit in which all three quarks are light compared with their chemical potential μ. In this regime the BCS instability leads to breaking of global (axial SU(3) and baryon-number U(1)), as well as local (color) symmetries. This phenomenon known as *color–flavor locking* (CFL) is an interesting example of a mechanism by which chiral symmetry can be broken spontaneously within the domain of weak-coupling perturbation theory. Since the pattern of breaking in CFL is somewhat similar to the vacuum of QCD, the lowest excitations – Goldstone bosons – are also similar, since their existence is dictated by the symmetries of the theory and the Goldstone theorem. What is new in CFL is that the properties of the Goldstone particles are calculable. Imagine calculating the pion decay constant from the first principles in QCD. This turns out to be possible in the CFL phase!

The high-temperature and high-chemical-potential environments will emphasize the property of asymptotic freedom of QCD, the fact that it is weakly coupled and perturbative at short distances while being strongly coupled and full of interesting nonperturbative symmetry realizations at large distances. One of the goals of lattice-gauge theory is to attain an understanding, both theoretical and numerical, of the physics on the multiple length scales of QCD: weak-coupling perturbative behavior at short distances, semi-classical nonperturbative behavior at intermediate distances, and strong-coupling confinement and color screening at large distances. We will cover the basics of lattice-gauge theory, with a strong emphasis on models in lower dimensions and spin models of greater simplicity, in order to develop the tools, insight, and experience to grapple with these phenomena in realistic four-dimensional settings. Duality, topological excitations and related order–disorder phase transitions, transfer matrices, and Hamiltonian methods will all be studied, with the goal of providing the reader with the skills to attack modern problems in QCD from several different vantage points.

A major development in modern field theory has been an understanding of the subtleties of fermion fields and their symmetries. This book spends considerable time on this subject since the lattice has played an important role in many of these topics. The most significant accomplishment of lattice-gauge theory was the demonstration that gauge theories could be formulated nonperturbatively. This means that local color symmetries could be defined even in an ultraviolet-regulated spacetime, and local gauge symmetries would remain exact. The important ingredient in this advance was the realization that the gauge group, rather than the generators, would provide the building blocks of the theory. The lattice action, for example, is constructed using a spacetime lattice of points and links, and gauge fields are represented by group elements on links and Fermi fields by Grassmann variables on sites. The notion of exact local gauge invariance, namely that the local color of an excitation tranforms under local color rotations and that the physics should be expressible in terms of invariants, has exact, geometrical expressions in this framework. This feature lies at the heart of the theory's ability to discuss confinement outside of particular calculational schemes and to discuss other associated nonperturbative phenomena quantitatively. Within this context one also discusses fermions and their special symmetries, such as flavor transformations and axial flavor and axial singlet transformations. This leads to nonperturbative formulations and calculations of chiral-symmetry breaking and dynamical mass generation. All of this is important to the phenomenology of QCD and its relation to the very successful constituent-quark model. The mechanisms of chiral-symmetry breaking in ordinary strongly interacting physics as well as in extreme environments of high temperature and chemical potentials have been revealed by lattice and semi-classical studies after decades of little progress when the only tool one had was perturbation theory. The interplay of chiral-symmetry breaking, semi-classical excitations, and confinement itself is a major theme in QCD and this book.

It is particularly interesting that these two successes, exact gauge invariance and chiral symmetries of fermions, lead to a near catastrophe in the subject. Naively it appears that global fermion symmetries can be formulated exactly on spacetime lattices. At the same time local gauge interactions can also be formulated exactly. This then presents the opportunity to "gauge" the fermion chiral symmetries and make models in which any global chiral symmetry of interest and any local gauge symmetry would also be exact, but such a "success" is too much of a good thing. We know from perturbation theory that axial symmetries are broken by the perturbative anomalies of gauge fields. For example, the neutral pion does decay into two photons and this is caused by the existence of the famous triangle graphs coupling a pion to two photons. In more basic terms, this result means that the axial flavor current that describes the neutral pion is no longer conserved in the presence of a quantized electromagnetic field. Classically there is a conservation law but quantum fluctuations destroy it. No one can argue that the neutral pion's dominant decay mode is into a state of two photons and few physicists would want to say that the standard formulation of axial symmetries and their currents does not describe the properties of light quarks and composite pions. It looks as though lattice-gauge theory can make quantum theories with too much symmetry!

Actually this problem is "solved" by another lattice shortcoming. If one considers the Dirac equation on the lattice, one finds surprises. In particular, there are typically too many species predicted from the discrete-lattice version of the Dirac equation in the theory's continuum limit and these extra species have just the right handedness and number to cancel out the anomaly that leads to the nonconservation of the global chiral current. This is the famous, or infamous, problem of "species doubling" on the lattice. It shows that the continuum limit of the lattice Dirac operator describes more species of fermions with more global symmetries than intended. In the early days of lattice-gauge theory this restriction led to formulations of lattice fermions that eliminated the unwanted extra fermion species by explicitly breaking most of the chiral symmetries. However, the breaking was constructed such that the continuum limit of the model would have the desired fermion species, gauge invariance, and anomalies, so that no contradictions could be encountered. This book will spend considerable space on these topics because they are central to understanding QCD. The two most useful lattice-fermion methods, which compromise chiral symmetries on the lattice but are engineered to have the correct continuum limits, are called "Wilson" and "staggered" fermions. They will be discussed in detail because they are easy to use both analytically and numerically. These formulations of the Dirac equation couple only nearest-neighbor sites on the lattice, and their actions can be "improved" in the sense that next-nearest-neighbor terms can be added to the action, so that the deviations of the lattice action from the continuum action can be systematically reduced. These improvements should prove important in numerical work on the phases of QCD.

The unpleasant feature of these lattice-fermion methods, namely that chiral symmetries are compromised on the cutoff system while gauge symmetries are not, is addressed in more sophisticated lattice-fermion actions. In particular, we will introduce the domain-wall fermion method which extends the lattice to five dimensions. In the new dimension there is a domain wall where massless, chiral fermions can exist. This scheme produces a four-dimensional lattice-gauge theory with full chiral symmetry, the correct anomaly structure and local gauge invariance. Another, closely related approach to this problem in clashing symmetries changes the definition of lattice chiral transformations to accomodate the nonzero lattice-spacing cutoff. In this scheme, Ginsparg–Wilson fermions, the cutoff system has an exact chiral-like symmetry and all of its important physical consequences but the price one pays here is that the continuous symmetry differs from the continuum chiral symmetry by lattice terms and there is apparent nonlocality in the four-dimensional action.

Our interest in these new lattice-fermion developments is restricted to their impact on understanding the nonperturbative, dynamical symmetry-breaking physics of QCD, which is a left–right-symmetric theory. It is hoped that the methods will produce useful nonperturbative formulations of the left–right-asymmetric models, such as the standard model of electroweak interactions, and other chiral gauge theories.

As we discuss these issues, we will introduce lattice-numerical methods with an emphasis on algorithms that can solve fermion field theories nonperturbatively. Recent developments in the field at high temperature and low chemical potential and their relevance to relativistic heavy-ion colliders (RHICs) will be emphasized.

Since lattice-gauge theory is not covered in graduate-physics curricula, we spend considerable time with the fundamentals of this subject. In fact, the chapters on lattice-gauge theory here could be used as the student's first introduction to the subject. The perspective is toward QCD in extreme environments, but all the fundamentals such as local gauge invariance, continuum limits and asymptotic freedom, confinement mechanisms, lattice fermions, computer algorithms, and chiral-symmetry breaking are covered here in an elementary and self-contained fashion. The use of model lattice-gauge-theory systems, field theories in two and three dimensions, occurs throughout the book to illustrate and introduce concepts with a minimum of formalism. Several sections of the book within the lattice-gauge-theory mantra give background to the goals of lattice-gauge theory by working through continuum-model field theories. For example, the importance of confinement, topology, the anomaly and $U(1)_A$ problem, and the existence of θ vacua are introduced and illustrated within $1 + 1$ quantum electrodynamics. This development emphasizes the need for lattice fermions to capture the chiral (flavor-singlet) and gauge symmetries of QCD correctly – if a failure occurred here, extra unphysical massless modes would populate the theory's physical spectrum. The importance of topological excitations in field theories is illustrated through the two-dimensional planar spin model, the Abelian Higgs model, and the O(3) model.

Much of the material presented here is adopted from the senior author's elementary reviews of the subject that were written when lattice-gauge theory was in its infancy [1].

The lattice approach to QCD provides a microscopic, fundamental formulation of the theory of quarks and gluons. The emphasis is on local gauge symmetries, their expression at the microscopic level, and their implications at large distances, as well as global symmetries, such as flavor and chiral symmetries. Eventually lattice QCD will provide concise solutions for the physics on all length scales. At this time, however, the field proceeds more modestly and in this book, for example, we supplement the lattice approach with the study of effective Lagrangians. In this approach one embodies the local and global symmetries of QCD in an effective field theory that states how the low-energy excitations of the solution of QCD must interact on the basis of our knowledge of the microscopic physics and the realizations of its fundamental symmetries. The successes here include an understanding of Goldstone and pseudo-Goldstone physics of pions, kaons, etc. Since light degrees of freedom control the critical behavior of QCD at its second-order phase transitions, this approach has much to say about extreme environments. In particular, at high temperature it predicts the critical behavior of the quark–gluon-plasma transition. Its generalization to nonzero chemical potential is also constrained by symmetries and conservation laws and will be developed in detail. We will see that it is particularly illuminating for model systems, such as SU(2) color models, in which the transition to a diquark condensate occurs at $\mu_c = m_\pi/2$, within the domain of applicability of chiral-perturbation theory. Unfortunately, it has less to predict about SU(3) QCD in cold but baryon-rich environments. A theoretical framework for QCD at baryon chemical potentials of the order of the mass of the nucleon is lacking, as will be discussed at length in the chapters that follow. One of the challenges here is to find a quantitative, nonperturbative framework to describe diquark condensation leading to breaking of color symmetry and color superconductivity.

One of the main themes of the book is to emphasize the use of methods that, starting from the defining principles of QCD as a quantum-field theory, are able to provide systematically controllable results. Historically, only perturbation theory lived up to this claim. Although the small-coupling expansion provides indispensable information, its domain of validity ends before, and often well before, the regime of interest, where nonperturbative phenomena such as confinement, chiral-symmetry breaking, and phase transitions set in. Lattice-field theory is an instrument that allows one to go into these domains with precision controlled by the lattice spacing. Hence, lattice-field theory takes a prominent place in this book. The method of effective Lagrangians is another very powerful tool, which allows one to embody concisely the information about the global symmetries of the theory and of the ground state. A systematically controllable expansion, now organized by powers of the small energies and momenta of the lowest excitations, can then provide nontrivial dynamical information, which in some cases is even exact. Sometimes the synonym *model-independent* is used to distinguish such systematically controlled, or exact, results from the results obtained using phenomenologically or theoretically motivated models. Such models have made and continue to make significant contributions to our qualitative and semiquantitative understanding of nonperturbative phenomena. Unfortunately, or perhaps fortunately, for the coming generation of researchers, at present, there is still a large domain of the QCD phase diagram that is not amenable to first-principles approaches. It is our belief that this will change and we hope that this book will help in that.

In writing a book of this sort it is hard to know where to start and where to stop. We assume that the reader has had or is taking an introductory field-theory course, so that path integrals, gauge fields, Feynman diagrams, and the quark model have been seen before. In addition, we use the language and methods of statistical physics, so some preparation in critical phenomena, Landau mean-field theory, and scaling laws would be useful. For example, we will freely move between the languages of field theory and statistical mechanics and we will assume that the reader can easily grasp the formal correspondences between the two subjects. We cover them briefly in our discussions of the transfer matrix for model lattice systems. Throughout the text we refer the reader to textbooks and review articles for additional background. The student might be studying introductory field theory or critical phenomena concurrently with a reading of this more specialized book. The student need not be an expert in these subjects when he or she begins this book, but we hope that they will emerge an expert at the end.

2

Background in spin systems and critical phenomena

2.1 Notation and definitions and critical indices

The language and formalisms of lattice-gauge theory are based on those of traditional statistical mechanics. Let's begin by introducing terminology and ideas in the context of the statistical mechanics of two-dimensional spin systems. This will lay the basis for more elaborate and challenging constructions in four-dimensional lattice-gauge theory.

Consider the two-dimensional Ising model on a square lattice whose sites are labeled by a vector of integers

$$n = (n_1, n_2) \tag{2.1}$$

Place a "spin" variable $s(n)$ at each site and suppose that s can have only two possible values: "up," $s = +1$, and "down," $s = -1$. Nearest-neighbor spins are defined to interact through an energy

$$S = -J \sum_{n,i} s(n)s(n+i) \tag{2.2}$$

where i denotes one of the two unit vectors of the square lattice, as shown in Fig. 2.1. We choose the coupling J to be positive, so that aligned spins are favored. In the context of Euclidean field theory, the energy of the system would be called the "action." This terminology will be clarified as we progress from statistical mechanics, with its partition function, order parameters, and susceptibilities, to Euclidean field theory, with its path integral, condensates, and propagators.

The Ising-model variables and energy have two particularly important features. First, the energy has only short-ranged interactions. In fact, the interactions involve only nearest-neighbor spins. The locality of the interactions will play a crucial role in determining the phase diagram and the critical behavior,