1
General Introduction

Whosoever loveth instruction loveth knowledge. . . .
Prov. xii 1

In this chapter the text begins with an informal introduction to the concept of stability and the nature of instability of a particular flow as a prototype – the flow along a pipe. The prototype illustrates the importance of instability as a prelude to transition to turbulence. Finally, the chief methods of studying instability of flows are briefly introduced.

1.1 Prelude

Hydrodynamic stability concerns the stability and instability of motions of fluids.

The concept of stability of a state of a physical or mathematical system was understood in the eighteenth century, and Clerk Maxwell (see Campbell & Garnett, 1882, p. 440) expressed the qualitative concept clearly in the nineteenth:

> When . . . an infinitely small variation of the present state will alter only by an infinitely small quantity the state at some future time, the condition of the system, whether at rest or in motion, is said to be stable; but when an infinitely small variation in the present state may bring about a finite difference in the state of the system in a finite time, the condition of the system is said to be unstable.

So hydrodynamic stability is an important part of fluid mechanics, because an unstable flow is not observable, an unstable flow being in practice broken down rapidly by some ‘small variation’ or another. Also unstable flows often evolve into an important state of motion called *turbulence*, with a chaotic three-dimensional vorticity field with a broad spectrum of small temporal and spatial scales called *turbulence*.

The essential problems of hydrodynamic stability were recognized and formulated in the nineteenth century, notably by Helmholtz, Kelvin, Rayleigh and Reynolds. It is difficult to introduce these problems more clearly than in Osborne Reynolds’s (1883) own description of his classic series of experiments on the instability of flow in a pipe, that is to say, a tube (see Figure 1.1 for

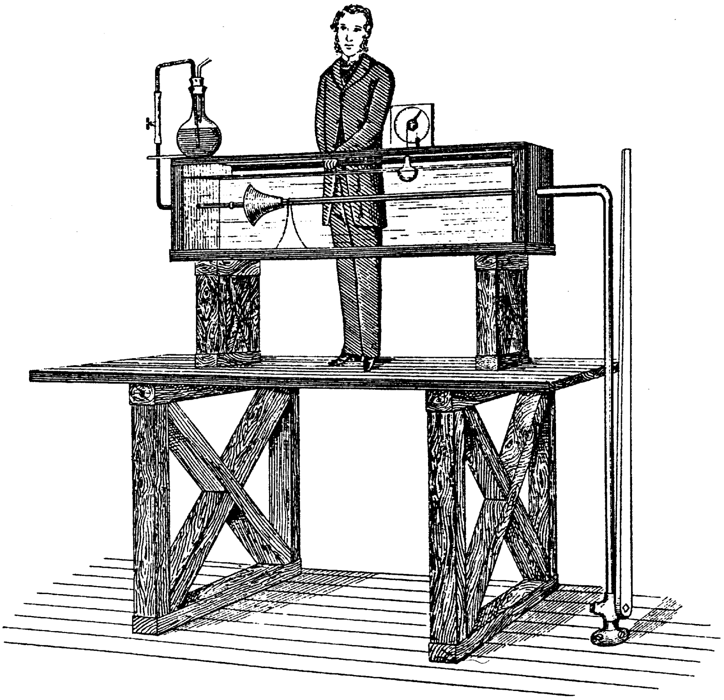

Figure 1.1 The configuration of Reynolds's experiment on flow along a pipe. (From Reynolds, 1883, Fig. 13.)

the general configuration of his apparatus, with an unnamed Victorian man to scale it).

The . . . experiments were made on three tubes . . . . The diameters of these were nearly 1 inch, $\frac{1}{2}$ inch and $\frac{1}{4}$ inch. They were all . . . fitted with trumpet mouthpieces, so that the water might enter without disturbance. The water was drawn through the tubes out of a large glass tank, in which the tubes were immersed, arrangements being made so that a streak or streaks of highly coloured water entered the tubes with the clear water.

The general results were as follows:–

(1) When the velocities were sufficiently low, the streak of colour extended in a beautiful straight line through the tube, Figure 1.2(a).

(2) If the water in the tank had not quite settled to rest, at sufficiently low velocities, the streak would shift about the tube, but there was no appearance of sinuosity.

(3) As the velocity was increased by small stages, at some point in the tube, always at a considerable distance from the trumpet or intake, the colour band would all at once mix up with the surrounding water, and fill the rest of the tube with a mass of coloured water, as in Figure 1.2(b).

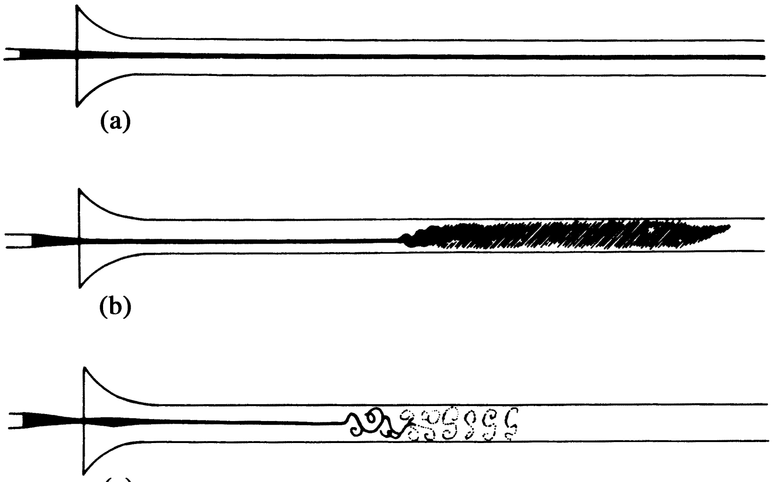

Figure 1.2 Sketches of (a) laminar flow in a pipe, indicated by a dye streak; (b) transition to turbulent flow in a pipe; and (c) transition to turbulent flow as seen when illuminated by a spark. (From Reynolds, 1883, Figs. 3, 4 and 5.)

Any increase in the velocity caused the point of break down to approach the trumpet, but with no velocities that were tried did it reach this.

On viewing the tube by the light of an electric spark, the mass of colour resolved itself into a mass of more or less distinct curls, showing eddies, as in Figure 1.2(c).

Reynolds went on to show that the *laminar flow*, the smooth flow he described in paragraph (1), breaks down when Va/ν exceeds a certain critical value, V being the maximum velocity of the water in the pipe, a the radius of the pipe, and ν the kinematic viscosity of water at the appropriate temperature. This dimensionless number Va/ν, now called the *Reynolds number*, specifies any class of dynamically similar flows through a pipe; here we shall denote the number by R. The series of experiments gave the critical value R_c of the Reynolds number as nearly 13 000. However,

the critical velocity was very sensitive to disturbance in the water before entering the tubes . . . .

This at once suggested the idea that the condition might be one of instability for disturbance of a certain magnitude and [stability] for smaller disturbances.

Just above the critical velocity

Another phenomenon . . . was the intermittent character of the disturbance. The disturbance would suddenly come on through a certain length of the tube and pass away

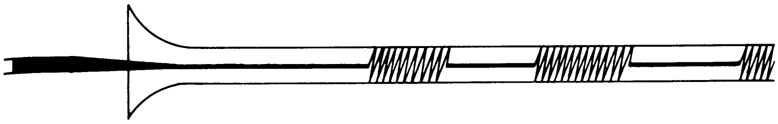

Figure 1.3 Crude sketch of turbulent spots in a pipe. (From Reynolds, 1883, Fig. 16.)

and then come on again, giving the appearance of flashes, and these flashes would often commence successively at one point in the pipe. The appearance when the flashes succeeded each other rapidly was as shown in Figure 1.3.

Such 'flashes' are now called *turbulent spots* or *turbulent bursts*. Below the critical value of the Reynolds number there was laminar Poiseuille pipe flow with a parabolic velocity profile, the resistance of the pipe (that is, the tube) to the flow of water being proportional to the mean velocity. As the velocity increased above its critical value, Reynolds found that the flow became *turbulent*, with a chaotic three-dimensional motion that strongly diffused the dye throughout the water in the pipe. The resistance of the pipe to turbulent flow grew in proportion to the square of the mean velocity.

Reynolds's original apparatus survives in Manchester in England, and was used in the 1970s to repeat his experiment. You can therefore see (Van Dyke, 1982, Fig. 103) photographs of the flow in Reynolds's apparatus.

Later experimentalists have introduced perturbations, that is to say, disturbances, of finite amplitude at the intake or used pipes with roughened walls to find R_c as low as 2000, and have used such regular flows and such smooth-walled pipes that R_c was 10^5 or even more. Reynolds's description illustrates the aims of the study of hydrodynamic stability: to find whether a given laminar flow is unstable and, if so, to find how it breaks down into turbulence or some other laminar flow.

Methods of analysing the stability of flows were formulated in Reynolds's time. The method of normal modes for studying the oscillations and instability of a dynamical system of particles and rigid bodies was already highly developed. A known solution of Newton's or Lagrange's equations of motion for the system was perturbed. The equations were linearized by neglecting products of the perturbations. It was further assumed that the perturbation of each quantity could be resolved into independent components or modes varying with time t like e^{st} for some constant s, which is in general complex. The values of s for the modes were calculated from the linearized equations. If the real part of s was found to be positive for any mode, the system was deemed unstable because a general initial small perturbation of the system would grow exponentially in time until it was no longer small. Stokes, Kelvin and Rayleigh adapted this method of normal modes to fluid dynamics. An essential mathematical difference between fluid and particle dynamics is that the equations of motion are partial rather than ordinary differential equations. This difference leads to many technical difficulties in hydrodynamic stability, which, to this day, have been fully overcome for only a few classes of flows with simple configurations.

Indeed, Reynolds's experiment itself is still imperfectly understood (Eliahou *et al.*, 1998). However, we can explain qualitatively the transition from laminar flow to turbulence with some confidence. Poiseuille pipe flow with a parabolic profile is stable to infinitesimal perturbations at all Reynolds numbers. At sufficiently small values of the Reynolds number, for $R \le R_g$, say, all perturbations, large as well as small, of the parabolic flow decay eventually; observation shows that $R_g \approx 2000$. Some way below the observed critical Reynolds number, a perturbation may grow if it is not too small. Above the critical Reynolds number quite small perturbations, perhaps introduced at the inlet or by an irregularity of the wall of the tube, grow rapidly with a sinuous motion. Soon they grow so much that nonlinearity becomes strong and large eddies (Figure 1.2(c)) or turbulent spots (Figure 1.3) form. (This mechanism, whereby a flow which is stable to all infinitesimal perturbations is made to change abruptly to a turbulent or nearly turbulent flow by a finite-amplitude perturbation, is now often called *bypass transition*.) As the Reynolds number increases, the threshold amplitude of perturbations to create instability decreases. At high Reynolds numbers turbulence ensues at once due to the inevitable presence of perturbations of small amplitude, and the flow becomes random, strongly three-dimensional (that is, very non-axisymmetric), and strongly nonlinear everywhere.† This instability of Poiseuille pipe flow may be contrasted with that of plane Poiseuille flow, which is unstable to infinitesimal perturbations at sufficiently large values of the Reynolds number. This explanation is supported by the treatment of the theory of the linear stability of Poiseuille pipe flow in §8.10. However, in practice the instability of plane Poiseuille flow resembles the instability of Poiseuille pipe flow, at least superficially (see Figure 1.4).

The physical mechanisms of Reynolds's experiments on instability of Poiseuille flow in a pipe are vividly illustrated by a film loop made by Stewart (FL1968) for the Education Development Center. This loop consists of edited excerpts from his longer film on *Turbulence* (Stewart, F1968). Details of these and other motion pictures on hydrodynamic stability may be found after the list of references at the end of the book. Videos of the experiment can be seen

† Many of the features of the transition from laminar to turbulent flow can easily be appreciated by observing the smoke from a cigarette. Light the cigarette, point the burning tip upwards, and observe the smoke as it rises from rest. See also Van Dyke (1982, Fig. 107).

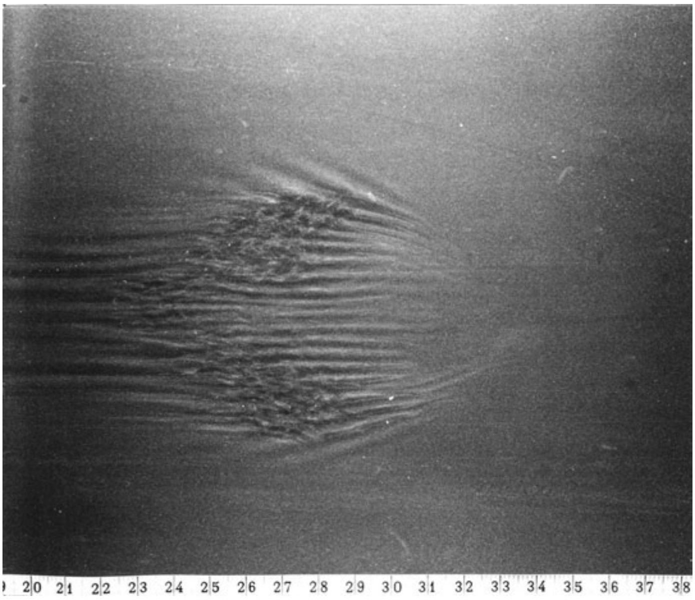

Figure 1.4 A turbulent spot triggered by jets in the wall of plane Poiseuille flow at $R = 1000$, where $R = Vd/\nu$, V is the maximum velocity of the flow, and the walls are separated by a distance $2d$. (From Carlson *et al.*, 1982, Fig. 4.)

by use of the compact disk of Homsy *et al.* (CD2000); this CD is currently more readily available than the film loops or their video versions, although briefer. Under the heading *Video Library* and subheadings 'Reynolds Transition Apparatus' and 'The Reynolds Transition Experiment', some short videos of recent experiments on Reynolds's original apparatus are shown; further experiments can be found under the subheadings 'Pipe Flow', 'Tube Flow' and 'Turbulent Pipe Flow'. Under the heading *Boundary Layers* and subheadings 'Instability, Transition and Turbulence' and 'Instability and Transition in Pipe and Duct Flow' more short videos are available.

1.2 The Methods of Hydrodynamic Stability

It may help at the outset to recognize that hydrodynamic stability has a lot in common with stability in many other fields, such as magnetohydrodynamics, plasma physics, elasticity, rheology, combustion and general relativity. The physics may be very different but the mathematics is similar. The mathematical essence is that the physics is modelled by nonlinear partial differential equations and the stability of known steady and unsteady solutions is examined. Hydrodynamics happens to be a mature subject (the Navier–Stokes equations having been discovered in the first half of the nineteenth century), and a given motion of a fluid is often not difficult to produce and to see in a laboratory, so hydrodynamic stability has much to tell us as a prototype of nonlinear physics in a wider context.

We learn about instability of flows and transition to turbulence by various means which belong to five more-or-less distinct classes:

(1) *Natural phenomena and laboratory experiments*. Hydrodynamic instability would need no theory if it were not observable in natural phenomena, man-made processes, and laboratory experiments. So observations of nature and experiments are the primary means of study. All theoretical investigations need to be related, directly or indirectly, to understanding these observations. Conversely, theoretical concepts are necessary to describe and interpret observations.
(2) *Numerical experiments*. Computational fluid dynamics has become increasingly important in hydrodynamic stability since 1980, as numerical analysis has improved and computers have become faster and gained more memory, so that the Navier–Stokes equations may be integrated accurately for more and more flows. Indeed, computational fluid dynamics has now reached a stage where it can rival laboratory investigation of hydrodynamic stability by simulating controlled experiments.
(3) *Linear and weakly nonlinear theory*. Linearization for small perturbations of a given basic flow is the first method to be used in the theory of hydrodynamic stability, and it was the method used much more than any other until the 1960s. It remains the foundation of the theory. However, weakly nonlinear theory, which builds on the linear theory by treating the leading nonlinear effects of small perturbations, began in the nineteenth century, and has been intensively developed since 1960.
(4) *Qualitative theory of bifurcation and chaos*. The mathematical theory of differential equations shows what flows *may* evolve as the dimensionless parameters, for example the Reynolds number, increase. The succession of bifurcations from one regime of flow to another as a parameter increases cannot be predicted quantitatively without detailed numerical calculations, but the admissible and typical routes to chaos and thence turbulence may be identified by the qualitative mathematical theory. Thus the qualitative theory of dynamical systems, as well as weakly nonlinear analysis, provides a useful conceptual framework to interpret laboratory and numerical experiments.

(5) *Strongly nonlinear theory*. There are various mathematically rigorous methods, notably Serrin's theorem and Liapounov's direct method, which give detailed results for arbitrarily large perturbations of specific flows. These results are usually bounds giving sufficient conditions for stability of a flow or bounds for flow quantities.

The plan of the book is to develop the major concepts and methods of the theory in detail, and then apply them to the instability of selected flows, relating the theoretical to the experimental results. This plan is itemized in the list of contents. First, in this and the next chapter, many concepts and methods will be described, and illustrated by simple examples. Then, case by case, these methods and concepts, together with some others, will be used in the later chapters to understand the stability of several important classes of flows. The theory of hydrodynamic stability has been applied to so many different classes of flow that it is neither possible nor desirable to give a comprehensive treatment of the applications of the theory in a textbook. The choice of applications below is rather arbitrary, and perhaps unduly determined by tradition. However, the choice covers many useful and important classes of flow, and illustrates well the five classes of general method summarized above.

1.3 Further Reading and Looking

It may help to read some of the following books to find fuller accounts of many points of this text. Many of the books are rather out of date, being written before the advent of computers had made much impact on the theory of hydrodynamic stability. (Perhaps computational fluid dynamics has led to the most important advances in recent years, and perhaps the theory of dynamical systems or applications of the theory has led to a wider physical range of new problems.) However, the subject is an old one, with most of the results of enduring importance, so these books are still valuable.

Betchov & Criminale (1967) is a monograph largely confined to the linear theory of the stability of parallel flows, covering numerical aspects especially well. Chandrasekhar (1961) is an authoritative treatise, a treasure house of research results of both theory and experiment. It emphasizes the linear stability of flows other than parallel flows, with influence of exterior fields such as magnetohydrodynamic, buoyancy and Coriolis forces. Its coverage of the literature is unusual, informative and of great interest. Drazin & Reid (1981) is a monograph with a broad coverage of the subject. It has several problems for students, but few of them are easy. Huerre & Rossi (1998) is a set of 'lecture notes', though at an appreciably higher level than this book. It is an account, mostly of linear stability of mostly parallel flows, with good modern coverage of numerical and experimental as well as theoretical results. Joseph (1976) is a monograph which emphasizes nonlinear aspects, especially the energy method, but has a broad coverage of basic flows. Landau & Lifshitz (1987) is a great treatise masquerading as a textbook; it summarizes the physical essentials of hydrodynamic stability with masterly brevity. Lin (1955) is a classic monograph, largely confined to the linear stability of parallel flows of a viscous fluid, the complement of Chandrasekhar's treatise. Schmid & Henningson (2001) is an up-to-date comprehensive research monograph on instability and transition of parallel flows.

We have already referred to pictures to enrich understanding of Reynolds's experiment. Such pictures are, of course, as valuable in the understanding of many other hydrodynamic instabilities. Van Dyke (1982) is a beautiful collection of photographs of flows, including hydrodynamic instabilities. Nakayama (1988) is another fine collection of photographs of flows, including hydrodynamic instabilities. Look at the photographs relevant to hydrodynamic stability, think about them, and relate them to the theory of this book. However, hydrodynamic instability is a dynamic phenomenon, best seen in motion pictures. So, many relevant films, film loops and videos, and the compact disk of Homsy *et al*. (CD2000), are listed in the Motion Picture Index at the end of the list of references. It is appropriate to add some words of caution here. The results of visualization of *un*steady flows are liable to be misinterpreted. Be careful. In particular, make sure that you understand the difference between streamlines, streaklines and particle paths before you jump to too many conclusions.

2

Introduction to the Theory of Steady Flows, Their Bifurcations and Instability

> ... whosoever heareth these sayings ..., and doeth them, ... will liken ... unto a wise man, which built his house upon a rock: And the rain descended, and the floods came, and the wind blew, and beat upon that house; and it fell not: for it was founded on rock.
>
> *Matt. viii 24–25*

The essences of the common forms of bifurcation, that is, the common types of change of regime of flow, are introduced in this chapter by use of simple illustrative ordinary differential problems. It is shown afterwards that these bifurcations occur where instability occurs. Finally, stability of a flow is defined mathematically, and the linearized problem and the method of normal modes are described.

2.1 Bifurcation

Consider flows of an incompressible viscous fluid in a given domain $\mathcal{V}$. Let ρ be the density of the fluid, and ν the kinematic viscosity. Let $\mathbf{u}_*$, p_* be the velocity and pressure of the fluid at a given point $\mathbf{x}_*$ at time t_*. Then flow is governed by the Navier–Stokes equations,

$$\frac{\partial \mathbf{u}_*}{\partial t_*} + \mathbf{u}_* \cdot \nabla_* \mathbf{u}_* = -\frac{1}{\rho} \nabla_* p_* + \nu \Delta_* \mathbf{u}_*,$$

and the equation of continuity,

$$\nabla_* \cdot \mathbf{u}_* = 0,$$

in $\mathcal{V}$; and certain boundary conditions, say

$$\mathbf{u}_* = \mathbf{U}_{0*} \quad \text{on part of } \partial\mathcal{V},\ \mathbf{u}_* \text{ is periodic on the rest of } \partial\mathcal{V};$$

where Δ_* is the Laplacian operator, $\partial\mathcal{V}$ is the boundary of $\mathcal{V}$ and $\mathbf{U}_{0*}$ is a given velocity of the fluid on the boundary.

Suppose that these equations and boundary conditions have a certain solution, approximate if not exact, which describes a steady flow whose stability is of