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Continuous Lattices and Domains

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Continuous Lattices and Domains

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Preface

BACKGROUND. In 1980 we published *A Compendium of Continuous Lattices*. A continuous lattice is a partially ordered set characterized by two conditions: firstly, completeness, which says that every subset has a least upper bound; secondly, continuity, which says that every element can be approximated from below by other elements which in a suitable sense are much smaller, as for example finite subsets are small in a set theoretical universe. A certain degree of technicality cannot be avoided if one wants to make more precise what this "suitable sense" is: we shall do this soon enough. When that book appeared, research on continuous lattices had reached a plateau.

The set of axioms proved itself to be very reasonable from many viewpoints; at all of these aspects we looked carefully. The theory of continuous lattices and its consequences were extremely satisfying for order theory, algebra, topology, topological algebra, and analysis. In all of these fields, applications of continuous lattices were highly successful. Continuous lattices provided truly interdisciplinary tools.

Major areas of application were the theory of computing and computability, as well as the semantics of programming languages. Indeed, the order theoretical foundations of computer science had been, some ten years earlier, the main motivation for the creation of the unifying theory of continuous lattices. Already the *Compendium of Continuous Lattices* itself contained signals pointing future research toward more general structures than continuous lattices. While the condition of *continuity* was a robust basis on which to build, the condition of completeness was soon seen to be too stringent for many applications in computer science – and indeed also in pure mathematics; an example is the study of the set of nonempty compact subsets of a topological space partially ordered by $\supseteq$: this set is a very natural object in general topology but fails to be a complete lattice in a noncompact Hausdorff space, while a filter basis of compact sets does have a nonempty intersection. Some form of completeness therefore should be retained; the form that is satisfied in most applications is that of "directed completeness", saying that every subset in which any two element set has an upper bound has a least upper bound; the existence of either a minimal or a maximal element is not implied.

In computer science it has become customary to speak of a poset with this weak completeness property as a deeceepea-oh, written **dcpo** (for **d**irected **c**omplete **p**artially **o**rdered set). *A continuous* **dcpo** *is what we call a* **domain**. Since this word appears in the title of this book, our terminology must be stated clearly at the beginning. In that branch of order theory with which this book deals there is no terminology clouded in more disagreement and lack of precision than that of a "domain", because it has become accepted as a sort of *nontechnical* terminology.

Domains in our sense had moved into the focus of researchers' attention at the time when the *Compendium of Continuous Lattices* was written, although then they were consistently called *continuous posets*, notably in the *Compendium* itself where they appear in many exercises. When their significance was discovered, it was too late to incorporate an emerging theory in the main architecture of the book, and it was too early for presenting a theory *in statu nascendi*. So we opted at that time for giving the reader an impression of things to come by indicating most of what we knew at the time in the form of exercises. The rising trend and our perception of it were confirmed in monographs, proceedings, and texts which appeared in a steady stream trailing the *Compendium*:

1981 Bernhard Banaschewski and Rudolf-Eberhard Hoffmann, editors, *Continuous Lattices*, Springer Lecture Notes in Mathematics **871**, x+413pp.,

1982 Rudolf-Eberhard Hoffmann, editor, *Continuous Lattices and Related Topics*, Mathematik Arbeitspapiere der Universität Bremen **27**, vii+314pp.,

1982 Peter Johnstone, *Stone Spaces*, Cambridge Studies in Advanced Mathematics **3**, xxi+370pp.,

1984 H. Lamarr Bentley, Horst Herrlich, M. Rajagopalan and H. Wolff, editors, *Categorical Topology*, Heldermann Sigma Series in Pure Mathematics **5**, xv+635pp.,

1985 Rudolf-Eberhard Hoffmann and Karl Heinrich Hofmann, editors, *Continuous Lattices and Their Applications*, Marcel Dekker Lecture Notes in Pure and Applied Mathematics **101**, x+369 pp.,

1989 Steven Vickers, *Topology via Logic*, Cambridge Tracts in Theoretical Computer Science **5** (2nd edition 1990), xii+200pp., 1990 B. A. Davey and H. A. Priestley, *Introduction to Lattices and Order*, Cambridge University Press, 1990, vii+248pp.,

1994 S. Abramsky and A. Jung, Domain theory, in S. Abramsky, D. M. Gabbay, and T. S. E. Maibaum, editors, *Handbook of Logic in Computer Science, Vol. III: Semantic Structures*, Oxford University Press,

1994 V. Stoltenberg-Hansen, I. Lindström, and E. R. Griffor, *Mathematical Theory of Domains*, Cambridge Tracts in Theoretical Computer Science **22**, xii+349pp.,

1998 R. M. Amadio and P.-L. Curien, *Domains and Lambda-Calculi*, Cambridge Tracts in Theoretical Computer Science **46**, xvi+484pp.

While some of these sources are devoted to supplementing the lattice theoretical and topological aspects of continuous lattices, the development of a more general domain theory and its computer theoretical applications predominate in this literature. From the viewpoint of pure mathematics, arguably the most prominent developments after the appearance of the *Compendium of Continuous Lattices* were

- the Lawson duality of domains (much in the spirit of Pontryagin duality of locally compact abelian groups),
- the first creation of a really satisfactory general theory of locally compact spaces in general topology via domain theory,
- other expanded connections with topology such as the the theory of sober spaces, principally the machinery surrounding the Hofmann–Mislove Theorem,
- the cross connections of domain theory and the theory of cartesian closed categories,
- the representation of topological spaces as the "ideal" or maximal points of a domain,
- and entirely new outlooks on classical analysis through domain theory.

AIMS. The *Compendium* by Gierz *et al.*, as it became known after a while, was out of print in a few years. It continued to be cited as a comprehensive reference on continuous lattices and their generalizations in spite of the cumbersome reference to a line of no less than six authors whose collaboration – notwithstanding their motley mathematical origin – was amply explained in the foreword of the *Compendium*; the five authors who had to take cover behind the hedge of "*et al.*" learned to live in hiding. The list of books which followed the *Compendium* is impressive. But somehow it seemed that the *Compendium* was not replaced or superseded, certainly not by one single book which could substitute for its expository and pedagogical drift. People felt that an attempt to overhaul the *Compendium* and to present a new edition containing the original information as well as reflecting developments of two decades of research in the larger scope of domain theory might be welcomed by readers in the area, old and young. In the fenced-in area of continuous lattices, the *Compendium* still had encyclopedic aspirations. As the vast literature of the last twenty years beyond the already respectable list of references in the *Compendium* indicates, this ambition is now beyond our grasp. It is therefore with a touch of modesty that, in the title of our book, we now drop the word *Compendium* and simply present a treatise on "Continuous Lattices and Domains".

As was its predecessor, this book is intended to present the mathematical foundations of the theory of continuous lattices and domains from the ingredients of order theory, topology and algebra and blends of all of these. Our use of category theory remains close to the concrete categories arising in our investigations, and thus we avoid the high degree of abstraction that category theory allows. It has been our deliberate choice only to lay the groundwork for the numerous applications that the theory of domains has found in the area of abstract theories of computation, the semantics of programming languages, logic and lambda calculus, and in other branches of mathematics. In the following selective list of subject matter not treated in this book, the reader may find guidance to further sources which are concerned with these and other applications; this list is far from being comprehensive.

- Domains for semantics of lambda calculi and of programming languages (see e.g. [Scott, 1993], [Scott, 1972a], [Scott, 1976], [Plotkin, B1981], [Gunter, B1992], [Winskel, B1993], [Amadio and Curien, B1998], [Reynolds, B1998]),
- stable domain theory, Girard's coherent spaces, hypercoherences (see e.g. [Amadio and Curien, B1998], [Girard, B1989], [Ehrhard, 1993]),
- Scott's information systems and more generally domain theory in logical form (see e.g. [Scott, 1982c], [Abramsky, 1991b], [Jung *et al.*, 1997]),
- domains and computability, computable analysis (see e.g. [Eršov, 1972a], [Stoltenberg-Hansen *et al.*, B1994], [Escardó, 1996a], [Edalat and Sünderhauf, 1999]),
- quantitative domain theory with its many different approaches,
- categorical generalizations (see e.g. [Adámek, 1997]),
- axiomatic and synthetic domain theory (see e.g. [Hyland, 1991], [Fiore, 1997], [Fiore and Rosolini, 1997a], [Fiore and Plotkin, 1997], [Taylor, B1998]),
- applications of domain theory in classical mathematics (see [Edalat, 1997a]).

GENESIS. In the foreword to the *Compendium* we familiarized the reader with some of the background and how it was written in a time that pre-dated the actual advent of TEX as the standard for mathematical typesetting. The writing of the new book proceeded under different auspices.

As a first step, even prior to our decision to go ahead with a new printing of the compendium, Dana S. Scott secured a complete LATEX source file of the *Compendium* in its entirety at Carnegie Mellon University; this source file was kept and elaborated typographically at the Technical University of Darmstadt in the custody of Klaus Keimel. We kept a pretty good record of all typographical and mathematical errors that we and our readers found in the *Compendium*, and all of these were corrected in our master file. A first updating of the bibliography of the *Compendium* was compiled by Rudolf-Eberhard Hoffmann, Karl Heinrich Hofmann, and Dana S. Scott in 1985 and was published in the Marcel Dekker volume edited by Hoffmann and Hofmann in 1985, pp. 303–360. At a time when we seriously thought about updating our data base on the literature for this book, an electronic file of the Marcel Dekker bibliography could no longer be located. Therefore this data base had to be reconstructed, and that was done under the supervision of Klaus Keimel at Darmstadt. In 2000 he also initiated a compilation of more current literature; many people contributed to that collection; we express our gratitude to all of them. Much of this material, although not all of it, entered the bibliography of this book. An Appendix to the *Compendium* (pp. 347–349) contained a listing on 52 Memos written and circulated in the Seminar on Continuity in Semilattices (SCS) from January 1976 through June 1979, because this body of material constituted much of the history of the content of the *Compendium*. The seminar continued for a number of years through June 1986, and we include in the present book a complete list of 98 SCS Memos (see pp. 564–567).

Several visits of Dana S. Scott's to Darmstadt consolidated the plan to envisage a new edition of the *Compendium*. Yet it became obvious soon that a considerable workload of rewriting would have to be done on the existing master file in order to accommodate domain theory. For any number of reasons it was not easy to get the project on its way; one of the simplest explanations is that the mathematical biographies of all of us had diverged sufficiently that the intensive spirit of collaboration of the 1970s was almost impossible to rekindle. Yet serious planning was undertaken by Hofmann and Lawson at a meeting at Louisiana State University at Baton Rouge on March 10, 2000, by Gierz, Hofmann, Keimel, and Lawson on March 16, 2000, at a workshop organized by the University of Riverside in honor of Albert Stralka on the occasion of his sixtieth birthday, and at a meeting of Klaus Keimel and Dana S. Scott on March 22, 2000, in Pittsburgh at Carnegie Mellon University. After these initiatives the actual rewriting began in earnest at Tulane University in New Orleans, at the Technical University of Darmstadt, and at Louisiana State University. It was helpful that a conference in Cork (Ireland) in July 2000 united Keimel, Lawson, Mislove, and Scott for discussions.

With respect to the writing itself, Chapters O, I, and II were revised jointly by Hofmann, Lawson, and Keimel, Chapters III, VI, and VII by Lawson, and Chapters IV and V by Keimel. The revisions consisted primarily of reformulating and supplementing from the lattice context to the **dcpo** context, a task that frequently proved nontrivial. In addition several new sections were written to reflect some of the developmental highlights since the *Compendium* appeared: Section O-5 on T_0 spaces (Lawson and Keimel), Sections III-3 and III-5 on quasicontinuous and compact domains (Lawson), Section IV-2 on duality (Hofmann), Sections IV-5 and IV-7 on pro-continuous functors and domain equations (Keimel), Sections IV-8 and IV-9 on powerdomains (Lawson), Section V-6 on domain environments (Lawson), Section VI-6 on stably compact spaces (Lawson), and Section VII-3 on hypercontinuity (Lawson). In addition Keimel prepared the comprehensive index and other end material.

HIGHLIGHTS. It is an indication of the robust architecture of the old *Compendium* that the actual rewriting could proceed largely by retaining the chapter subdivision and revising and amplifying the old content. However, **dcpo**s replaced complete lattices wherever possible from Chapter O on and domains replaced continuous lattices where possible. This was not always possible; a good example is what we used to call "the algebraic characterization of continuous lattices" in the *Compendium*. This is attached to the monadic character of continuous lattices and simply fails for domains. Occasionally a good deal of work had to be invested to accommodate the more general viewpoint.

Chapter II on the Scott topology of domains is a case in point. We have amplified the function space aspect, described the Isbell topology on function spaces, and exposed it as a true generalization of the classical compact open topology. Furthermore we discuss the poset $Q(X)$ of compact saturated sets on a T_0 space with respect to the partial order $\supseteq$, allowing a full treatment of the Hofmann–Mislove Theorem and its various aspects and exposing some new aspects. We also elaborate on certain cartesian closed categories of domains.

Chapter III elaborates on what is known on the Lawson topology on domains and their compactness properties for this topology and thus contains much information that was not present in the *Compendium*. In the section on "Quasicontinuity and Liminf Convergence" we introduce a class of posets containing that of domains properly and call its members *quasicontinuous domains*. On a quasicontinuous domain, the Scott topology discussed in Chapter II is locally compact and sober, and the Lawson topology (the main topic of Chapter III) is regular and Hausdorff, and indeed much of the theory of domains can be recovered in this more general setting. A notion of liminf convergence is introduced, which is shown to be equivalent to topological convergence in the Lawson topology for (quasicontinuous) domains. In the old *Compendium* the concept of "quasicontinuity" was called the "GCL-property", as in "generalized continuous lattice". The section entitled "Compact Domains" is largely devoted to the question of when the Lawson topology on a domain is compact and wraps up with the theory of the Isbell topology in the context of function space topologies.

Beyond that which Chapter IV contained in the *Compendium*, it now presents a full treatment of the attractive Lawson duality of domains, which parallels Pontryagin duality – notably when it is restricted to the category of continuous semilattices (that is, domains which in addition are inf semilattices) and Scott-continuous semilattice morphisms; in that form it is a veritable character theory for domains. The Lawson duality of continuous semilattices allows us to round off the complex of the Hofmann–Mislove Theorem which was presented in Chapter II. A sort of geometric aspect of the duality between two domains is exposed in Chapter V, because it realizes a pair of dual domains as the spectrum and the cospectrum of a completely distributive complete lattice.

The section on projective limits in Chapter IV is now formulated for the category of domains (or even **dcpo**s) and morphisms appearing as pairs of a Galois adjunction; in the case of maps between complete lattices preserving arbitrary infs or sups this is automatic; in the more general setup of domains the presence of Galois adjunctions must be postulated. Wherever we had operated in the *Compendium* on a largely category theoretical level, we do not have to adjust our approach fundamentally to make it work in the more general **dcpo** framework that interests us in this book. However, the more general and updated treatment of these matters has resulted in a considerable expansion of these sections. The chapter closes with an introduction to the important topic of powerdomains, including the extended probabilistic powerdomain.

Chapter V in the *Compendium* dealt with the spectral theory of continuous lattices. Since spectral theory is largely a formalism applying to lattices, this chapter has remained largely stable, but it was augmented by a section on *domain environments* which illustrates a novel application of domain theory to that branch of analysis dealing largely with Polish spaces. It is in the nature of some of the material in the *Compendium* that it is not or only marginally affected by the general upgrading from continuous lattices to domains; sections dealing with such material remain preserved in the way they were in the old *Compendium*.

In Chapter VI the reader will find a new section, in which under the heading "Stably Compact Spaces" we discuss a concept of compact spaces which emulate in the wide class of T_0 spaces as many properties as seem reasonable of classical compact T_2 spaces. These spaces have a partner topology, called the co-compact topology, and the common refinement, called the patch topology, is a compact Hausdorff topology. The most prominent example of a stably compact space is a domain with the Scott topology such that the Lawson topology is compact; in this example the patch topology is the Lawson topology. We close Chapter VI with the spectral theory of these spaces.

Chapter VII includes a new section on "Hypercontinuity and Quasicontinuity". Hypercontinuous lattices are a special class of continuous lattices for which, among several diverse characterizations, the interval topology is Hausdorff. They stand in spectral duality to the quasicontinuous domains equipped with the Scott topology.

NOTES. The notes at the end of each section make some attempt to relate the material to the published literature, but these references are only representative, not comprehensive. Subsections entitled "Old Notes" are largely duplicated from *A Compendium of Continuous Lattices*, except for an effort to accommodate any renumbering that has taken place. Since individual contributions could at that time be identified via SCS Memos, which are listed in the bibliography, and since such a multiplicity of authors was involved, it seemed reasonable to depart from traditional practice and more or less identify some of the major contributions of various authors in the notes. Subsections entitled "New Notes" have been added to those sections that are new or significantly different from those appearing in the *Compendium*. Thus sections with little or no revision may have only "Old Notes", new sections will have only "New Notes", and old sections with significant revisions will have both.

BIBLIOGRAPHY. The literature about domain theory and continuous lattices has grown to such proportions that a comprehensive bibliography is not feasible. We have tried, however, to compile an extensive bibliography relevant to the topics treated in this book. The bibliography is subdivided into several sections:

- books, monographs, and collections, cited in the form [Gierz *et al.*, B1980], where the B refers to a book,
- conference proceedings, cited in the form [1982, Bremen] giving the year and the place of the conference,

- articles, cited simply in the form [Abramsky and Jung, 1994],
- dissertations and master's theses, cited in the form [Lawson, D1967], where the D refers to a dissertation,
- SCS Memos, cited in the form [SCS 15].

MIZAR. This is also the place to report on an activity of the MIZAR project group located primarily at the University of Białystok, Poland, the University of Alberta, Edmonton, Canada, and the Shinshu University, Nagano, Japan. It is the aim of the MIZAR project to codify mathematical knowledge in a data base. The codification means the formalization of concepts and proofs mechanically checked for logical correctness. The MIZAR language is a formal language derived from the mathematical vernacular. The principal idea was to design a language that is readable by mathematicians, and simultaneously, is sufficiently rigorous to enable processing and verifying by computer software.

Our monograph *A Compendium of Continuous Lattices* was chosen by the MIZAR group for testing their system. Since 1995, the *Compendium* has been translated piece by piece into the language MIZAR. As of August 2002, sixteen authors have worked on this specific project; they have produced fifty-seven MIZAR articles. The work is still in progress. For details one may consult the MIZAR homepage ⟨http://www.mizar.org/⟩ and the report on the work concerning the *Compendium* ⟨http://megrez.mizar.org/ccl/⟩.

The Authors
January, 2002

Acknowledgments

We thank Cambridge University Press for publishing this book and its referees for having scrutinized the project and recommended publication. David Tranah of CUP has been particularly helpful in communications and organization. Many persons interested in domain theory encouraged us to go ahead with the project of presenting the *Compendium of Continuous Lattices* in an updated version to the public.

At Carnegie-Mellon University, Staci Quackenbush carried out the task of creating a LaTeX file of the old *Compendium*. At the Technical University of Darmstadt several people worked on this master file of the old *Compendium* by proofreading, inserting pictures and diagrams, notably Michal Konečny, and Michael Marz, who were funded as Wissenschaftliche Hilfskräfte am Fachbereich Mathematik der TUD. Cathy Fischer helped with these files in May 2000 and was funded by SEFO – Frauenselbsthilfe und Fortbildungszentrum e.V. in Darmstadt. Our special thanks go to Thomas Erker for his major contributions in creating the appropriate macro apparatus for indexing, in setting up the bibliography, and in making the whole file system function smoothly.

Andrej Bauer created for us an electronic archive, at which the master copies of the main files were eventually kept. The archive made it possible for us to keep our sanity while several people were working on the same portions of the book at different locations around the globe. Diagrams were drawn with the aid of Paul Taylor's package.

Several universities and agencies supported extended periods of concentrated work on the manuscript. In February and March 2001, Klaus Keimel visited Tulane University and was funded by a grant from the Office of Naval Research to Michael Mislove. In summer 2001 he spent one month at the University of Birmingham on a visiting position of the Computer Science Department; he profited in his work on this monograph from the advice of Achim Jung and Martín Escardó. Karl H. Hofmann received travel support from the Tulane Mathematics Department to spend time at Tulane on various occasions. Jimmie Lawson visited the Technische Universität Darmstadt for two weeks during the summer of 2001 and for a month during the winter of 2002, the second visit being funded by the Alexander von Humboldt Foundation.

Foreword to *A Compendium of Continuous Lattices*

A mathematics book with six authors is perhaps a rare enough occurrence to make a reader ask how such a collaboration came about. We begin, therefore, with a few words on how we were brought to the subject over a ten-year period, during part of which time we did not all know each other. We do not intend to write here the history of continuous lattices but rather to explain our own personal involvement. History in a more proper sense is provided by the bibliography and the notes following the sections of the book, as well as by many remarks in the text. A coherent discussion of the content and motivation of the whole study is reserved for the introduction.

In October of 1969 Dana Scott was led by problems of semantics for computer languages to consider more closely partially ordered structures of function spaces. The idea of using partial orderings to correspond to spaces of partially defined functions and functionals had appeared several times earlier in recursive function theory; however, there had not been very sustained interest in structures of continuous functionals. These were the ones Scott saw that he needed. His first insight was to see that – in more modern terminology – the category of algebraic lattices and the (so-called) Scott-continuous functions is cartesian closed. Later during 1969 he incorporated lattices like the reals into the theory and made the first steps toward defining continuous lattices as "quotients" of algebraic lattices. It took about a year for the topological ideas to mature in his mind culminating in the paper published as [Scott, 1972a]. (For historical points we cannot touch on in this book the reader is referred to Scott's papers.) Of course, a large part of Scott's work was devoted to a presentation of models for the type-free lambda-calculus, but the search for such models was not the initial aim of the investigation of partially ordered structures; on the contrary, it was the avoiding of the formal and unmotivated use of lambda-calculus that prompted Scott to look more closely at the structures of the functions themselves, and it was only well after he began to see their possibilities that he realized there had to exist nontrivial T_0 spaces homeomorphic to their own function spaces.

Quite separately from this development, Karl Hofmann, Jimmie Lawson, Mike Mislove, and Al Stralka (among others) recognized the importance of compact semilattices as a central ingredient in the structure theory of compact semigroups. In his dissertation [D1967], Lawson initiated the study of a class of compact semilattices distinguished by the property that each had enough continuous semilattice morphisms into the unit interval semilattice (in its natural order) to separate points. (Such a program had already been started by Nachbin for partially ordered spaces in [Nachbin, B1965].) Lawson characterized this class of compact semilattices as those which admitted a basis of subsemilattice neighborhoods at each point (small subsemilattices): the class proved to be of considerable theoretical interest and attracted the attention of other workers in the field. In fact, it was believed for some time that all compact semilattices were members of the class, partly because the theory was so satisfactory (for example, purely "order theoretic" characterizations were discovered for the class by [Lawson, 1973]), and because no natural counterexamples seemed to exist. However, Lawson found the first example of a compact semilattice which was not in the class, one in fact which admitted only constant morphisms into the unit interval [Lawson, 1970] (see Chapter VI, Section 4).

At about the same time, Klaus Keimel had been working on lattices and lattice ordered algebras in pursuit of their spectral theory and their representation in sheaves. In his intensive collaboration with Gerhard Gierz on topological representations of nondistributive lattices, a spectral property emerged which turned out to be quite significant for compact semilattices with small subsemilattices.

The explanation for the fact that the topological algebra of Lawson's semilattices had been so satisfactory emerged clearly when Hofmann and Stralka gave a completely lattice theoretical description of the class [Hofmann and Stralka, 1976]. It was Stralka who first recognized the relation of this class to Scott's continuous lattices, and this observation came about as follows. Two monographs on duality theories for lattices and topological structures emerged in the early seventies: One for topology and lattices by Hofmann and Keimel [B1972], and the other for compact zero dimensional semilattices and lattices by Hofmann, Mislove, and Stralka [B1974]. At the lattice theory conference in Houston in 1973, where such dualities were discussed, B. Banaschewski spoke on filters and mentioned Scott's work which was just about to appear in the *Proceedings of the Dalhousie Category Theory Conference*. Stralka checked out this hint, and while he and Hofmann were working on the algebraic theory of Lawson semilattices [Hofmann and Stralka, 1976], he realized the significance of this work as a link between the topological algebra of compact semilattices and the lattice theory of Scott's continuous lattices. This led to correspondence with Scott and much subsequent activity.

In the summer of 1976, Hofmann and Mislove spent some time collaborating with Keimel and Gierz at the Technische Hochschule in Darmstadt, and together they began a "write-in" seminar called the Seminar on Continuity in Semilattices, or SCS for short. The authors formed the core membership of the seminar, but their colleagues and students contributed greatly to the seminar by communicating their results, ideas, and problems. (A list of these seminar reports (SCS Memos) which resulted is provided at the end of this monograph.) The seminar then convened in person for several lively and well-attended workshops. The first was hosted by Tulane University in the spring of 1977, the second by the Technische Hochschule Darmstadt in the summer of 1978, and the third by the University of California at Riverside in the spring of 1979. A fourth workshop was held at the University of Bremen in the fall of 1979. We are very much indebted to all who participated in these seminars and others whose in fluence on this book is very considerable. In particular we thank H. Bauer, J. H. Carruth, Alan Day (who discovered an independent access to continuous lattices through the filter monad), Marcel Erné, R.-E. Hoffmann, John Isbell (who also gave very detailed remarks on the present manuscript), Jaime Ninio, A. R. Stralka, and O. Wyler.

It was at the Tulane Workshop that the idea of collecting together the results of research – common and individual – was first discussed. A preliminary version of the *Compendium* worked out primarily by Hofmann, Lawson, and Gierz was circulated among the participants of the Darmstadt Workshop, and many people gave us their useful reactions. For help in typing the earlier versions of this book we would like to thank Frau Salder in Darmstadt and Mrs. Meredith Mickel at Tulane University.

The preparation of the final version of the text, which is reproduced from camera-ready copy, was carried out by and under the direction of Scott at the Xerox Palo Alto Research Center (PARC) in its Computer Science Laboratory (CSL). Scott spent the academic year 1978/79 on sabbatical as a Visiting Scientist at Xerox PARC, and the facilities of CSL, including extra secretarial aid, were very generously put at his disposal. The text was prepared on an Alto computer using the very flexible BRAVO text-editing system and a special computer-controlled printer. The typist, who in the course of the project also became a skilled computer-aided book illustrator and copy editor, was Melinda Maggiani. Without her loyal efforts and concentrated labor the book would never have been put into anywhere near the form seen here; the authors are extremely grateful to her. Special thanks are also due to many members of CSL for their interest and patience in helping Scott learn to use BRAVO, with which he spent long, long hours; he wishes to mention with great warmth in particular Sara Dake, Leo Guibas, Jim Horning, Jeannette Jenkins, Joe Maleson, Jim Morris, HayChan Sargent, and Dan Swinehart. In the very last stages of the book make-up it was necessary to reprogram some printing routines to overcome several most irritating difficulties, and Lyle Ramshaw then stepped in and solved all the programming problems in record time. We take our hats off to him. (See especially in this regard Chapter IV, Section 3.)

The computerized editing system made it possible to produce in a very few months what were in effect two complete sets of galley proofs and two complete sets of page proofs; this is something that would never have been possible in our wildest dreams with the conventional manuscript–typescript–type style of book production. Computer-controlled editing allowed the authors to make, through the fingers of Maggiani and Scott, innumerable substantive corrections and to do extensive rewriting at every stage of the proofreading up to the last day before printing the camera-ready copy. Authors and publishers alike can only hope that such systems will soon become widely available. It was a real privilege to prepare this book at Xerox PARC, and the authors record here their heart-felt thanks to Dr. Robert J. Spinrad, Vice President and Manager of Xerox PARC, and especially to Robert Taylor, Manager of CSL. Aside from the support and cooperation, the remarkably friendly and informal atmosphere of PARC contributed much to the project.

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The Authors
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Introduction to *A Compendium of Continuous Lattices*

Background and Plan of the Work

The purpose of this monograph is to present a fairly complete account of the development of the theory of continuous lattices as it currently exists. An attempt has been made to keep the body of the text expository and reasonably self-contained; somewhat more leeway has been allowed in the exercises. Much of what appears here constitutes basic, foundational or elementary material needed for the theory, but a considerable amount of more advanced exposition is also included.

Background and Motivation

The theory of continuous lattices is of relatively recent origin and has arisen more or less independently in a variety of mathematical contexts. We attempt a brief survey in the following paragraphs in the hope of pointing out some of the motivation behind the current interest in the study of these structures. We first indicate a definition for these lattices and then sketch some ways in which they arise.

A DEFINITION. In the body of the *Compendium* the reader will find many equivalent characterizations of continuous lattices, but it would perhaps be best to begin with one rather straightforward definition – though it is not the primary one employed in the main text. Familiarity with *algebraic lattices* will be assumed for the moment, but even if the exact details are vague, the reader is surely familiar with many examples: the lattice of ideals of a ring, the lattice of subgroups of a group.

Abstractly (and up to isomorphism) we can say that an algebraic lattice is a lattice of sets – contained, say, in the lattice of all subsets of a given set A – closed under arbitrary intersection of families of sets and under unions of directed families of sets (e.g. chains of sets). These are important closure properties of the lattice of ideals, for example. If we think of the powerset lattice as a product 2^A of A copies of the two-element lattice $2 = \{0, 1\}$, then an algebraic lattice is just a *sublattice* of 2^A with respect to the infinite operations of arbitrary pointwise inf and pointwise sup of *directed* families of lattice elements. (Note, however, that *finite* sups are different in general; so the meaning of the word "sublattice" has to be understood in a suitable sense.)

Let us now replace the discrete lattice 2 with the "continuous" lattice $[0, 1]$, the unit interval of real numbers with its natural order and familiar lattice structure. In a power $[0, 1]^A$ we can speak of sublattices with respect to arbitrary pointwise inf and pointwise sup of directed families of elements, just as before. *Up to isomorphism, these are exactly the continuous lattices*. Of course this definition gives no hint as to the internal structure of these lattices and is only a dim indicator as to their naturalness and usefulness. But it does show that they are direct generalizations of well-known kinds of lattices and that they have an important element of "continuity".

THEORY OF COMPUTATION. Often in computational schemes one employs some algorithm successively to gain increasingly refined approximations to the desired result. It is convenient to use, formally or informally, topological language – one talks about "how far" the approximation is from the desired result or how good a "fit" has been obtained. An alternate procedure is to specify at each stage a subset in which the desired result lies. The smaller the set, the better the approximation; we could say that the smaller set gives "more information". This approach leads naturally to the use of order theoretic language in discussing the partial results, and the data generated, in a way related to the containments among the sets.

Let us now abstract this approach somewhat. Let P be a partially ordered set. We think of a "computation" of an element x in P as being a sequence of increasingly larger elements – "larger" meaning "more" in the sense of information – whose supremum is x. (More generally, we could imagine a directed set whose supremum is x.) We wish to regard x as the "limit" of the sequence (or set) of approximations.

What is needed is a precise definition of how well some "stage" of the "computation" approximates the "limit" x. We take an indirect approach to this question, because there is no metric available to tell us immediately how close an approximation is to the desired limit. We define in place of a metric a notion meaning roughly: an element y is a "finite approximation" to the element x. Then, to have a well-behaved system of limits, we *assume* that every element is the sup in the partial ordering of its finite approximations. A given sequence of approximations to x is then "successful" if it eventually encompasses all the finite approximations.

Specifically, we say that an element y, which is less than or equal to x, is a *finite approximation* to x if for any directed set D with supremum x (D represents the stages of a "computation" of x) we have some member of D which is greater than or equal to y. (Hence, all the members from that stage on are greater than or equal to y.) The idea is that if we use y to measure the accuracy of computations of x, then every computation that *achieves* x must eventually be *at least as accurate as* y.

Strictly speaking the outline just given is actually not quite right. We should say that if a directed set D has supremum *greater than or equal to* x, then some member of D is greater than or equal to y. This ensures that if y is a "finite approximation" for x, then it is also one for every element larger than x – a property we would certainly want to require. In the text we use different terminology. The notion of "finite" in "finite approximation" is somewhat vague, because again there is no measure to distinguish finite elements from infinite ones in general; indeed there are lattices where *all* elements except 0 are normally thought of as infinite (as in the lattice [0, 1] for instance). This explains our feeling that another terminology was required. We have used the phrase "y is way below x" for topological and order theoretic reasons cited in the appropriate section of the book.

To recapitulate: we assume that the "finite approximations" for each element are directed and have that element for their supremum. *The complete lattices with this property are the continuous lattices*. It is the theory of these abstracted, order theoretic structures that we develop in this monograph. It should be pointed out that only the lattice case is treated in the main text; generalizations appear in the exercises.

Owing to limitations of length and time, the theory of computation based on this approach is not developed extensively here. Certain related examples are, however, mentioned in the present text or in the exercises. For instance, consider the set of all partial functions from the natural numbers into itself (or some distinguished subset such as the recursive partial functions). These can be ordered by inclusion (that is, extension). Here again the larger elements give more information. In this example f is a "finite approximation" for g if and only if g is an extension of f and the domain of f is finite. In many examples such as this the "approximating" property can be interpreted directly as a finiteness condition, since there are finite functions in the set (functions with a finite domain). This circumstance relates directly to the theory of algebraic lattices, a theme which we do cover here in great detail.

GENERAL TOPOLOGY. Continuous lattices have also appeared (frequently in cleverly disguised forms) in *general topology*. Often the context is that of the category of all topological spaces or of topological spaces where one assumes only a T_0 separation axiom. Such spaces have been the objects of renewed interest with the emergence of spectral theory.

In fact, a continuous lattice can be endowed in a natural way with a T_0 topology which is defined from the lattice structure; in this book we call this T_0 topology the *Scott topology*. It is shown in Section 3 of Chapter II that these spaces are exactly the "injectives" (relative injectives in the categorical sense) or "absolute retracts" in the category of all T_0 spaces and continuous functions; that is if f is a continuous function from a subspace X of a topological space Y into a continuous lattice L (equipped with the Scott topology), then there always exists a continuous extension of f from X to Y with values still in L. This property in fact gives a topological characterization of continuous lattices, since any T_0 topology of such a space is just the Scott topology of a continuous lattice naturally determined from it.

In another direction let us say that an open set U is *relatively compact* in an open set V if every open cover of V has finitely many members which cover U. If X is a topological space, then the lattice of open sets is a continuous lattice iff each open set is the union of the open sets which are relatively compact in it. In this case the "way-below" relation is viewed as just the relation of one open set being relatively compact in another. This illustrates some of the versatility of the concept of a continuous lattice.

Spaces for which the lattice of open sets is a continuous lattice prove to be quite interesting. For Hausdorff spaces it is precisely the locally compact spaces which have this property, and in more general spaces analogs of this result remain true. We investigate this situation in some detail in the context of the spectral theory of distributive continuous lattices in Section 5 of Chapter V. It is often the case that theorems concerning locally compact Hausdorff spaces extend to spaces with a continuous lattice of open sets in the category of all topological spaces (see, e.g., [Day and Kelly, 1970]). Such considerations provide another link between continuous lattices and general topology.

The dual of the lattice of open sets – the lattice of closed sets – has long been an object of interest to topologists. If X is a compact Hausdorff space, then the lattice of closed subsets under the Vietoris topology is also a compact Hausdorff space. In Chapter III we introduce a direct generalization of this topology, called here the Lawson topology, which proves to be compact for all complete lattices and Hausdorff for continuous lattices. This connection allows applications of continuous lattice theory to the topological theory of hyperspaces (cf. Example VI-3.8).

ANALYSIS AND ALGEBRA. Several applications of continuous lattices arise in analysis and functional analysis. For example, consider the family $C(X, \mathbb{R})$ of continuous real-valued functions on the locally compact space X. Using the pointwise operations and the natural order from $\mathbb{R}$, the space $C(X, \mathbb{R})$ is a lattice, but its lattice theory is rather unsatisfactory. For example, it is not even complete; however, if we consider this lattice as a sublattice of $\mathrm{LSC}(X, \mathbb{R}^*)$, the lattice of all lower semicontinuous extended real-valued functions on X, then we do have a complete lattice with which to work. In fact, although this is not at all apparent from the functional analysis viewpoint, the lattice $\mathrm{LSC}(X, \mathbb{R}^*)$ is a continuous lattice. This entails several results, not the least of which is the following: *The lattice* $\mathrm{LSC}(X, \mathbb{R}^*)$ *admits a unique compact Hausdorff function-space topology such that* $(f, g) \mapsto f \wedge g$ *is a continuous operation*. (See I-1.22, II-4.7, and II-4.20.) In light of the fact that $C(X, \mathbb{R})$ is never compact and that $C(X, \mathbb{R}^*)$ is compact only if X is finite, this result is somewhat suprising; we do not know of a "classical" proof. Indeed, lower semicontinuity motivates one of the canonical topologies on any continuous lattice, and, if we equip $\mathbb{R}^*$ with this canonical topology, then the continuous functions from X into $\mathbb{R}^*$ so topologized are exactly those extended real-valued functions on X which are lower semicontinuous relative to the usual topology on $\mathbb{R}^*$. In the same vein it emerges that the probability distribution functions of random variables with values in the unit interval form a continuous lattice; compact topologies for this example are, however, familiar from classical analysis (cf. I-2.22).

A second example is quite different and demonstrates an overlap between analysis and algebra. With a ring R one associates a topological space, called its *spectrum*, and while there are many ways of doing this, probably the most wide spread and best known is the space of prime ideals of a commutative ring endowed with the hull–kernel topology. This plays a central role in algebraic geometry (where the relevant theory deals with noetherian rings and their spectra), and this construction can also be carried out for Banach algebras, in which case the preferred spectrum is the space of closed primitive ideals (which reduces to the more familiar theory of maximal ideals if the algebra is commutative). These particular ideals are relevant since they are precisely the kernels of irreducible representations of the algebra as an algebra of operators on a Banach space or Hilbert space.

Now, the connection of these spectral theories with the theory of continuous lattices emerges more clearly if we first return to the case of a commutative ring R. In this case, the spectrum is the set of prime ideals viewed as a subset of the algebraic lattice of all ideals of the ring; in fact, the spectrum is exactly the family of prime elements of the distributive algebraic lattice of all radical ideals of R. (Recall that a radical ideal is one which is the intersection of prime