O

A Primer on Ordered Sets and Lattices

This introductory chapter serves as a convenient source of reference for certain basic aspects of complete lattices needed in what follows. The experienced reader may wish to skip directly to Chapter I and the beginning of the discussion of the main topic of this book: continuous lattices and domains.

Section O-1 fixes notation, while Section O-2 defines complete lattices, complete semilattices and directed complete partially ordered sets (**dcpo**s), and lists a number of examples which we shall often encounter. The formalism of Galois connections is presented in Section 3. This not only is a very useful general tool, but also allows convenient access to the concept of a Heyting algebra. In Section O-4 we briefly discuss meet continuous lattices, of which both continuous lattices and complete Heyting algebras (frames) are (overlapping) subclasses. Of course, the more interesting topological aspects of these notions are postponed to later chapters. In Section O-5 we bring together for ease of reference many of the basic topological ideas that are scattered throughout the text and indicate how ordered structures arise out of topological ones. To aid the student, a few exercises have been included. Brief historical notes and references have been appended, but we have not tried to be exhaustive.

O-1 Generalities and Notation

Partially ordered sets occur everywhere in mathematics, but it is usually assumed that the partial order is *antisymmetric*. In the discussion of nets and directed limits, however, it is not always so convenient to assume this property. We begin, therefore, with somewhat more general definitions.

Definition O-1.1. Consider a set L equipped with a reflexive and transitive relation $\leq$. Such a relation will be called a *preorder* and L a *preordered set*. We say that a is a *lower bound* of a set $X \subseteq L$, and b is an *upper bound*, provided that

$$a \leq x \text{ for all } x \in X, \quad \text{and}$$
$$x \leq b \text{ for all } x \in X, \quad \text{respectively.}$$

A subset D of L is *directed* provided it is nonempty and every finite subset of D has an upper bound in D. (Aside from nonemptiness, it is sufficient to assume that every *pair* of elements in L has an upper bound in L.) Dually, we call a nonempty subset F of L *filtered* if every finite subset of F has a lower bound in F.

If the set of upper bounds of X has a unique smallest element (that is, the set of upper bounds contains exactly one of its lower bounds), we call this element the *least upper bound* and write it as $\bigvee X$ or $\sup X$ (for *supremum*). Similarly the *greatest lower bound* is written as $\bigwedge X$ or $\inf X$ (for *infimum*); we will not be dogmatic in our choice of notation. The notation $x = \bigvee^{\uparrow} X$ is a convenient device to express that, firstly, the set X is directed and, secondly, x is its least upper bound. In the case of pairs of elements it is customary to write

$$x \wedge y = \inf\{x, y\},$$
$$x \vee y = \sup\{x, y\}.$$

These operations are also often called *meet* and *join*, and in the case of meet the multiplicative notation xy is common and often used in this book. □

Definition O-1.2. A *net* in a set L is a function $j \mapsto x_j : J \to L$ whose domain J is a directed set. (Nets will also be denoted by $(x_j)_{j \in J}$, by (x_j), or even by x_j, if the context is clear.)

If the set L also carries a preorder, then the net x_j is called *monotone* (resp., *antitone*), if $i \leq j$ always implies $x_i \leq x_j$ (resp., $x_j \leq x_i$).

If $P(x)$ is a property of the elements $x \in L$, we say that $P(x_j)$ holds *eventually* in the net if there is a $j_0 \in J$ such that $P(x_k)$ is true whenever $j_0 \leq k$.

The next concept is slightly delicate: if L carries a preorder, then the net x_j is a *directed net* provided that for each fixed $i \in J$ one eventually has $x_i \leq x_j$. A *filtered net* is defined dually. □

Every monotone net is directed, but the converse may fail. Exercise O-1.12 illustrates pitfalls to avoid in defining directed nets. The next definition gives us some convenient notation connected with upper and lower bounds. Some important special classes of sets are also singled out.

Definition O-1.3. Let L be a set with a preorder $\leq$. For $X \subseteq L$ and $x \in L$ we write:

(i) $\downarrow X = \{y \in L : y \leq x \text{ for some } x \in X\}$;
(ii) $\uparrow X = \{y \in L : x \leq y \text{ for some } x \in X\}$;

(iii) $\downarrow x = \downarrow\{x\}$;
(iv) $\uparrow x = \uparrow\{x\}$.

We also say:

(v) X is a *lower set* iff $X = \downarrow X$;
(vi) X is an *upper set* iff $X = \uparrow X$;
(vii) X is an *ideal* iff it is a directed lower set;
(viii) X is a *filter* iff it is a filtered upper set;
(ix) an ideal is *principal* iff it has a maximum element;
(x) a filter is *principal* iff it has a minimum element;
(xi) $\mathrm{Id}\, L$ (resp., $\mathrm{Filt}\, L$) is the set of all ideals (resp. filters) of L;
(xii) $\mathrm{Id}_0\, L = \mathrm{Id}\, L \cup \{\emptyset\}$;
(xiii) $\mathrm{Filt}_0\, L = \mathrm{Filt}\, L \cup \{\emptyset\}$. □

Note that the principal ideals are just the sets $\downarrow x$ for $x \in L$. The set of lower bounds of a subset $X \subseteq L$ is equal to the set $\bigcap\{\downarrow x : x \in X\}$, and this is the same as the set $\downarrow \inf X$ in case $\inf X$ exists. Note, too, that

$$X \subseteq \downarrow X = \downarrow(\downarrow X),$$

and similarly for $\uparrow X$.

Remark O-1.4. *For a subset X of a preordered set L the following are equivalent:*

(1) *X is directed;*
(2) *$\downarrow X$ is directed;*
(3) *$\downarrow X$ is an ideal.*

Proof: (2) iff (3): By Definition O-1.3.

(1) implies (2): If A is a finite subset of $\downarrow X$, then there is a finite subset B of X such that for each $a \in A$ there is a $b \in B$ with $a \leq b$ by O-1.3(i). By (1) there is in X an upper bound of B, and this same element must also be an upper bound of A.

(2) implies (1): If A is a finite subset of X, it is also contained in $\downarrow X$; therefore, by (2), there is an upper bound $y \in \downarrow X$ of A. By definition $y \leq x \in X$ for some x, and this x is an upper bound of A. □

Remark O-1.5. *The following conditions are equivalent for L and X as in* O-1.4*:*

(1) $\sup X$ *exists;*
(2) $\sup \downarrow X$ *exists.*

And if these conditions are satisfied, then $\sup X = \sup \downarrow X$. *Moreover, if every finite subset of* X *has a sup and if* F *denotes the set of all those finite sups, then* F *is directed, and* (1) *and* (2) *are equivalent to*

(3) $\sup F$ *exists.*

Under these circumstances, $\sup X = \sup F$. *If* X *is nonempty, we need not assume the empty sup belongs to* F.

Proof: Since, by transitivity and reflexivity, the sets X and $\downarrow X$ have the same set of upper bounds, the equivalence of (1) and (2) and the equality of the sups are clear. Now suppose that $\sup A$ exists for every finite $A \subseteq X$ and that F is the set of all these sups. Since $A \subseteq B$ implies $\sup A \leq \sup B$, we know that F is directed. But $X \subseteq F$, and any upper bound of X is an upper bound of $A \subseteq X$; thus, the sets X and F have the same set of upper bounds. The equivalence of (1) and (3) and the equality of the sups is again clear, also in the nonempty case. □

The – rather obvious – theme behind the above remark is that statements about arbitrary sups can often be reduced to statements about finite sups and sups of directed sets. Of course, both O-1.4 and O-1.5 have straightforward duals.

Definition O-1.6. A partial order is a transitive, reflexive, and antisymmetric relation $\leq$. (This last means $x \leq y$ and $y \leq x$ always imply $x = y$.) A *partially ordered set*, or *poset* for short, is a nonempty set L equipped with a partial order $\leq$. We say that L is *totally ordered*, or a *chain*, if all elements of L are comparable under $\leq$ (that is, $x \leq y$ or $y \leq x$ for all elements $x, y \in L$). An *antichain* is a partially ordered set in which any two different elements are incomparable, that is, in which $x \leq y$ iff $x = y$. □

We have remarked informally on duality several times already, and the next definition makes duality more precise.

Definition O-1.7. For $R \subseteq L \times L$ any binary relation on a set L, we define the *opposite relation* R^{op} (sometimes: the *converse relation*) by the condition that, for all $x, y \in L$, we have $x R^{\mathrm{op}} y$ iff $y R x$.

If in $(L, \leq)$, a set equipped with a transitive, reflexive relation, the relation is understood, then we write L^{op} as short for $(L, \leq^{\mathrm{op}})$. □

The reader should note that if L is a poset or a chain, then so is L^{op}. One should also be aware how the passage from L to L^{op} affects upper and lower bounds. Similar questions of duality are also relevant to the next (standard) definition.

Definition O-1.8. An *inf semilattice* is a poset S in which any two elements a, b have an inf, denoted by $a \wedge b$ or simply by ab. Equivalently, a semilattice is a poset in which every nonempty finite subset has an inf. A *sup semilattice* is a poset S in which any two elements a, b have a sup $a \vee b$ or, equivalently, in which every nonempty finite subset has a sup. A poset which is both an inf semilattice and a sup semilattice is called a *lattice*.

As we will deal with inf semilattices very frequently, we adopt the shorter expression "semilattice" instead of "inf semilattice".

If a poset L has a greatest element, it is called the *unit* or *top* element of L and is written as 1 (or, rarely, as $\top$). The top element is the inf of the empty set (which, if it exists, is the same as $\sup L$). A semilattice with a unit is called *unital*. If L has a smallest element, it is called the *zero* or *bottom* element of L and is written 0 (or $\perp$). The bottom element is the sup of the empty set (which, if it exists, is the same as $\inf L$). □

Note that in a semilattice an upper set is a filter iff it is a subsemilattice. A dual remark holds for lower sets and ideals in sup semilattices. We turn now to the discussion of maps between posets.

Definition O-1.9. A function $f: L \to M$ between two posets is called *order preserving* or *monotone* iff $x \leq y$ always implies $f(x) \leq f(y)$. A one-to-one function $f: L \to M$ where both f and f^{-1} are monotone is called an *isomorphism*. We denote by *POSET* the category of all posets with order preserving maps as morphisms.

We say that f preserves

(i) *finite sups*, or (ii) (*arbitrary*) *sups*, or (iii) *nonempty sups*, or (iv) *directed sups*

if, whenever $X \subseteq L$ is

(i) finite, or (ii) arbitrary, or (iii) nonempty, or (iv) directed,

and $\sup X$ exists in L, then $\sup f(X)$ exists in M and equals $f(\sup X)$. A parallel terminology is applied to the preservation of infs. □

In the case of (iv) above, the choice of expression may not be quite satisfactory linguistically, but the correct phrase "preserves least upper bounds of directed sets" is too long. The preservation of directed sups can be expressed in the form

$$f\left(\bigvee^{\uparrow} X\right) = \bigvee^{\uparrow} f(X).$$

For semilattices a map preserving nonempty finite infs might be called a *homomorphism* of semilattices. The reader should notice that a function preserving *all* finite infs preserves the inf of the empty set; that is, it maps the unit to the unit – provided that unit exists. In order to characterize maps f preserving only the nonempty finite infs (if this is the condition desired), we can employ the usual equation:

$$f(x \wedge y) = f(x) \wedge f(y),$$

for $x, y \in L$. Note that such functions are monotone, and the dual remark also holds for homomorphisms of sup semilattices.

Remark. It should be stressed that our definition of "preservation of sups" is quite strong, as we require that, whenever a set X in the domain has a sup, then its image has a sup in the range. As a consequence, if a function $f: L \to M$ preserves (directed) sups, it also preserves the order. Indeed, if $a \leq b$ in L, then $\{a, b\}$ is a (directed) set that has a sup; as f preserves (directed) sups, then $f(a) \vee f(b)$ exists and $f(b) = f(a \vee b) = f(a) \vee f(b)$, whence $f(a) \leq f(b)$.

Often in the literature a weaker definition is adopted: f "preserves sups" if whenever $\sup X$ and $\sup f(X)$ both exist, then $f(\sup X) = \sup f(X)$. In this weak sense, a one-to-one map from the two element chain to two incomparable elements preserves sups. Thus a function that preserves (directed) sups in this weak sense need not be order preserving. In order to avoid ambiguities one should keep in mind that if a map preserves (directed) sups in our sense, then it is automatically order preserving. This implies in particular that the image of a directed set is also directed.

Remark O-1.10. *Let $f: L \to M$ be a function between posets. The following are equivalent:*

(1) *f preserves directed sups;*
(2) *f preserves sups of ideals.*

Moreover, if L is a sup semilattice and f preserves finite sups, then (1) *and* (2) *are also equivalent to*

(3) *f preserves arbitrary sups.*

A dual statement also holds for filtered infs, infs of filters, semilattices and arbitrary infs.

Proof: Both conditions (1) and (2) imply the monotonicity of f. Then the equivalence of (1) and (2) is clear from O-1.4 and O-1.5. Now suppose L is a sup semilattice and f preserves finite sups. Let $X \subseteq L$ have a sup in L. By the method of O-1.5(3), we can replace X by a directed set F having the same sup. Hence, if (1) holds, then $f(\sup X) = \sup f(F)$. But since f preserves finite sups, it is clear that $f(F)$ is constructed from $f(X)$ in the same way as F was obtained from X. Thus, by another application of O-1.5(3), we conclude that $f(\sup X) = \sup f(X)$. That (3) implies (1) is obvious. □

Exercises

Exercise O-1.11. Let $f: L \to M$ be monotone on posets L and M, and let $X \subseteq L$. Show that $\downarrow f(X) = \downarrow f(\downarrow X)$. □

Exercise O-1.12. Construct a net $(x_j)_{j \in J}$ with values in a poset such that for all pairs $i, j \in J$ there is a $k \in J$ with $x_i \leq x_k$ and $x_j \leq x_k$ but such that $(x_j)_{j \in J}$ is *not* directed.

Hint. Consider the lattice $2 = \{0, 1\}$, let $J = \{0, 1, 2, \ldots\}$, and let the net be defined so that $x_i = 0$ iff i is even. □

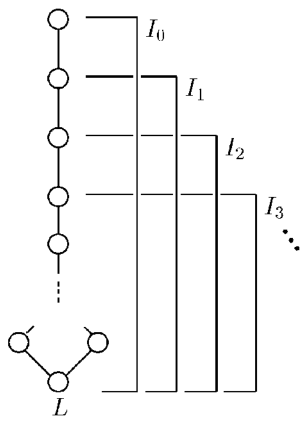
Exercise O-1.13. Modify O-1.10 so that for (3) we have only to assume that f preserves *nonempty* finite sups. □

Exercise O-1.14. Is the category of preordered sets and monotone maps *equivalent* to the category of posets and monotone maps? In these categories what sort of functor is $^{\text{op}}$? □

Exercise O-1.15. Let L be a poset, and let the I_j for $j \in J$ be ideals of L. Prove the following.

(i) $\bigcap_j I_j$ is an ideal of L iff $\bigcap_j I_j \neq \emptyset$, for L a sup semilattice.
(ii) In general, $\bigcap_j I_j$ is not necessarily an ideal of L, even if $\bigcap_j I_j \neq \emptyset$.

Hint. Consider the semilattice and ideals in the following figure.



(iii) The intersection $I_1 \cap I_2$ of two ideals I_1, I_2 is an ideal, for L a semilattice.
(iv) If L is directed, $\bigcup_j I_j$ is contained in some ideal of L (however, even if this is the case, there need not be a smallest ideal containing $I_1 \cup I_2$) and the converse holds if this is true for any two ideals I_1, I_2.
(v) Id L is a sup semilattice iff L is a sup semilattice.

Hint. If L is a sup semilattice, then $I = \downarrow\{a \vee b\colon a \in I_1, b \in I_2\}$ is the sup of the ideals I_1 and I_2 of L. Conversely, if Id L is a sup semilattice, then we claim there is a unique element $c \in \downarrow a \vee \downarrow b$ with $a, b \leq c$. Indeed, there is at least one since $\downarrow a \vee \downarrow b$ is directed; moreover, if c and c_1 were two such elements, then $\downarrow c$ and $\downarrow c_1$ would be two ideals of L both containing a and b and both contained in $\downarrow a \vee \downarrow b$. Hence $\downarrow c = \downarrow c_1 = \downarrow a \vee \downarrow b$.

(vi) Dual statements hold for Filt L, where one assumes L is a semilattice in part (v). □

Exercise O-1.16. Let L be a preordered set, and let $\mathcal{L}$ denote the family of all nonempty lower sets of L. Prove the following.

(i) Id $L \subseteq \mathcal{L}$ and $\mathcal{L}$ is a sup semilattice.
(ii) If L is a poset, then the map $x \mapsto \downarrow x\colon L \to \mathcal{L}$ is an isomorphism of L onto the family of principal lower sets of L.
(iii) If L is a filtered poset, then $\mathcal{L}$ is a lattice with respect to intersection and union.
(iv) Let L and M be semilattices, $f\colon L \to M$ be a function, and $\mathcal{L}$ and $\mathcal{M}$ be the lattices of nonempty lower sets. Let $f_* = (A \mapsto \downarrow f(A))\colon \mathcal{L} \to \mathcal{M}$. Then f is a semilattice morphism iff f_* is a lattice morphism. □

Old notes

The notion of a directed set goes back to the work of [Moore and Smith, 1922], where they use directed sets and nets to determine topologies. A convenient survey of this theory is provided in Chapter 2 of [Kelley, B1955]; we shall utilize this approach in our treatment of topologies on lattices, especially in Chapters II and III of this work. The material in this section is basic and elementary; a guide to additional reading – if more background is needed – is provided in the notes for Section O-2.

O-2 Completeness Conditions for Lattices and Posets

No excuse need be given for studying complete lattices, because they arise so frequently in practice. Perhaps the best infinite example (aside from the lattice of all subsets of a set) is the unit interval $\mathbb{I} = [0, 1]$. Many more examples will be found in this text – especially involving nontotally ordered lattices.

Definition O-2.1. (i) A poset is said to be *complete with respect to directed sets* (shorter: *directed complete* or also *up-complete*) if every directed subset has a sup. A **d***irected* **c***omplete* **po***set* is called a **dcpo** for short. A **dcpo** with a least element is called a *pointed* **dcpo**, or a **dcpo** *with zero* 0 or *with bottom* $\perp$.

(ii) A poset which is a semilattice and directed complete will be called a *directed complete semilattice*.

(iii) A *complete lattice* is a poset in which *every* subset has a sup and an inf. A totally ordered complete lattice is called a *complete chain*.

(iv) A poset is called a *complete semilattice* iff every nonempty (!) subset has an inf and every directed subset has a sup.

(v) A poset is called *bounded complete*, if every subset that is bounded above has a least upper bound. In particular, a bounded complete poset has a smallest element, the least upper bound of the empty set. □

We advise the reader to keep in mind that "up-complete poset" and "**dcpo**" are completely synonymous expressions; this advice is appropriate since the second terminology has become prevalent in the theoretical computer science community and since we use it in this book. We observe in the following that *a poset is a complete lattice iff it is both a* **dcpo** *and a sup semilattice with a smallest element.* In the exercises for this section we comment further on the relation of the concepts we have just introduced.

Proposition O-2.2. *Let L be a poset.*

(i) *For L to be a complete lattice it is sufficient to assume the existence of arbitrary sups (or the existence of arbitrary infs).*
(ii) *For L to be a complete lattice it is sufficient to assume the existence of sups of finite sets and of directed sets (or the existence of finite infs and filtered infs).*
(iii) *If L is a unital semilattice, then for completeness it is sufficient to assume the existence of filtered infs.*
(iv) *L is a complete semilattice iff L is a bounded complete* **dcpo**.

Proof: For (i) we observe that the existence of arbitrary sups implies the existence of arbitrary infs. Let $X \subseteq L$ and let

$$B = \bigcap\{\downarrow x \colon x \in X\}$$

be the set of lower bounds of X. (If X is empty, we take $B = L$.) We wish to show that

$$\sup B = \inf X.$$

If $x \in X$, then x is an upper bound of B; whence, $\sup B \leq x$. This proves that $\sup B \in B$; as it clearly is the maximal element of B, this also proves that X has a greatest lower bound. (There is obviously a dual argument assuming infs exist.)

For (ii) we first observe by Remark O-1.5 that the existence of finite sups and of directed sups implies the existence of arbitrary sups and then apply part (i).

For (iii), since the existence of finite infs is being assumed, the existence of all infs follows from (the dual of) (ii).

For a proof of (iv) if L is a complete semilattice and $A \subseteq L$ is bounded above, then the set of upper bounds has a greatest lower bound which will be the least upper bound of A. Conversely, for a bounded complete **dcpo** L and $\emptyset \neq A \subseteq L$ the 0 is contained in the set B of lower bounds of A. Any member of A is an upper bound of B and hence B has a least upper bound which is the greatest lower bound of A. □

Many subsets of complete lattices are again complete lattices (with respect to the restricted partial ordering). Obviously, if we assume that $M \subseteq L$ is *closed* under arbitrary sups and infs of the complete lattice L, then M is itself a complete lattice. But this is a very strong assumption on M. In view of O-2.2, if we assume only that M is closed under the sups of L, then M is a complete lattice (in itself as a poset). The well-worn example is with L equal to *all* subsets of a topological space X and with M the lattice of *open* subsets of X. This example is instructive because in general M is not closed under the infs of L (open sets are not closed under the formation of infinite intersections). Thus the infs of M (as a complete lattice) are *not* the infs of L. (**Exercise**: What is the simple topological definition of the infs of M?)

An even more general construction of subsets which form complete lattices is provided by the next theorem from [Tarski, 1955]. This theorem is of great interest in itself, as it implies that every monotone self-map on a complete lattice has a greatest fixed-point and a least fixed-point.

Theorem O-2.3. (The Tarski Fixed-Point Theorem) *Let $f: L \to L$ be a monotone self-map on a complete lattice L. Then the set $\mathrm{fix}(f) = \{x \in L : x = f(x)\}$ of fixed-points of f forms a complete lattice in itself. In particular, f has a least and a greatest fixed-point.* □