145 Isoperimetric Inequalities

This introduction treats the classical isoperimetric inequality in Euclidean space and contrasting rough inequalities in noncompact Riemannian manifolds. In Euclidean space the emphasis is on quantitative precision for very general domains, and in Riemannian manifolds the emphasis is on qualitative features of the inequality that provide insight into the coarse geometry at infinity of Riemannian manifolds.

The treatment in Euclidean space features a number of proofs of the classical inequality in increasing generality, providing in the process a transition from the methods of classical differential geometry to those of modern geometric measure theory; and the treatment in Riemannian manifolds features discretization techniques and applications to upper bounds of large time heat diffusion in Riemannian manifolds.

The result is an introduction to the rich tapestry of ideas and techniques of isoperimetric inequalities, a subject that has beginnings in classical antiquity and that continues to inspire fresh ideas in geometry and analysis to this very day – and beyond.

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Isoperimetric Inequalities

Differential Geometric and Analytic Perspectives

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Preface

This book discusses two venues of the isoperimetric inequality: (i) the sharp inequality in Euclidean space, with characterization of equality, and (ii) isoperimetric inequalities in Riemannian manifolds, where precise inequalities are unavailable but rough inequalities nevertheless yield qualitative global geometric information about the manifolds.

In Euclidean space, a variety of proofs are presented, each slightly more ambitious in its application to domains with irregular boundaries. One could easily go directly to the final definitive theorem and proof with little ado, but then one would miss the extraordinary wealth of approaches that exist to study the isoperimetric problem. An idea of the overwhelming variety of attack on this problem can be quickly gleaned from the fundamental treatise of Burago and Zalgaller (1988); and I have attempted on the one hand to capture some of that variety, and on the other hand to find a more leisurely studied approach that covers less material but with more detail.

In Riemannian manifolds, the treatment is guided by two motifs: (a) the dichotomy between the local Euclidean character of all Riemannian manifolds and the global geometric properties of Riemannian manifolds, this dichotomy pervading the study of nearly all differential geometry, and (b) the discretization of Riemannian manifolds possessing bounded geometry (some version of local uniformity). The dichotomy between local and global is expressed, in our context here, as the study of properties of Riemannian manifolds that remain invariant under the replacement of a compact subset of the manifold with another of different geometry and topology, as long as the new one fits smoothly in the manifold across the boundary of the deletion of the original compact subset. Thus, we do not seek fine results, in that we study coarse robust invariants that highlight the "geometry at infinity" of the manifold. Our choice of isoperimetric constants will even be invariant with respect to the discretization of the Riemannian manifold. The robust character of these new isoperimetric constants will then allow us to use this discretization to show how the geometry at infinity influences large time heat diffusion on Riemannian manifolds.

Regrettably, there is hardly any discussion of isoperimetric inequalities on compact Riemannian manifolds. That would fill a book – quite different from this one – all by itself.

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A summary of the chapters goes as follows:

Chapter I starts with posing the isoperimetric problem in Euclidean space and gives some elementary arguments toward its solution in the Euclidean plane. These arguments are essentially a warm-up. They are followed by a summary of background definitions and results to be used later in the book. Thus the discussion of the isoperimetric problem, proper, begins in Chapter II.

Chapter II starts with uniqueness theory, under the assumption that the boundary of the solution domain is C^2. We first show that, if a domain Ω with C^2 boundary is a solution to the isoperimetric problem for domains with C^2 boundaries, Ω must be an open disk. Then we strengthen the result a bit – we show that if a domain is but an extremal for isoperimetric problems, then it must be a disk. Then we consider the existence of a solution to the isoperimetric problem for domains with C^1 boundaries. We give M. Gromov's argument that for such domains the disk constitutes a solution to the isoperimetric problem. But only if one restricts oneself to convex domains with C^1 boundaries does his argument imply that the disk is the unique solution.

Chapter III is the heart of the first half of the book. It expands the isoperimetric problem in that it considers all compacta and assigns the Minkowski area to each compact subset of Euclidean space to describe the size of the boundary. In this setting, using the Blaschke selection theorem and Steiner symmetrization, one shows that the closed disk constitutes a solution to the isoperimetric problem. Since the Minkowski area of a compact domain with C^1 boundary is the same as the differential geometric area of the boundary, the result extends the solution of the isoperimetric problem from the C^1 category to compacta. Moreover, one can use the traditional calculations to show that the disk is the unique solution to the isoperimetric problem in the C^1 category. But uniqueness in the more general collection of compacta is too difficult for such elementary arguments.

Then, in Chapter III, we recapture Steiner's original intuition that successive symmetrizations could be applied to any compact set to ultimately have it converge to a closed disk – in the topology of the Hausdorff metric on compact sets. We use this last argument to prove the isoperimetric inequality for compacta with finite perimeter. The perimeter, as a measure of the area of the boundary, seems to be an optimal general setting, since one can not only prove the isoperimetric inequality for compacta with finite perimeter, but can also characterize the case of equality.

In Chapter IV we introduce Hausdorff measure for subsets of Euclidean space and develop the story sufficiently far to prove that the perimeter of a Lipschitz domain in n-dimensional Euclidean space equals the $(n-1)$-dimensional Hausdorff measure of its boundary. The proof involves the *area formula,* for which we include a proof.

Chapter V begins a new view of isoperimetric inequalities, namely, rough inequalities in a Riemannian manifold. The goal of Chapters V–VIII is to show how these geometric isoperimetric inequalities influence the qualitative rate of decay, with respect to time, of heat diffusion in Riemannian manifolds.

In Chapter V we summarize the basic notions and results concerning isoperimetric inequalities in Riemannian manifolds, and in Chapter VI we give their implications for analytic Sobolev inequalities on Riemannian manifolds. Chapter V consists, almost entirely, of a summary of results from my *Riemannian Geometry*: *A Modern Introduction,* and I have included just those proofs that seemed to be important to the discussion here. The discussion of Sobolev inequalities in Chapter VI has received extensive treatment in other books, but our interest is restricted to those inequalities required for subsequent applications. Moreover, we have also treated the relation of Sobolev inequalities on Riemannian manifolds and their discretizations, one to the other. To my knowledge, this has yet to be treated systematically in book form.

Chapter VII introduces the Laplacian and the heat operator on Riemannian manifolds and is devoted to setting the stage for the "main problem" in large time heat diffusion; its formulation and solution are presented in Chapter VIII. The book ends with an introduction to the new arguments of A.A. Grigor'yan, the full possibilities of which have only begun to be realized.

* * *

I have attempted to strike the right balance between merely summarizing background material (of which there is quite a bit) and developing preparatory arguments in the text. Also, although I have summarized the necessary basic definitions and results from Riemannian geometry at the beginning of Chapter V, I occasionally require some of that material in earlier chapters, and I use it as though the reader already knows it. This seems the lesser of two evils, the other evil being to disrupt the flow of the arguments in the first half of the book for an excursus that would have to be repeated in its proper context later. Most of that material is quite elementary and standard, so it should not cause any major problems.

In order to clarify somewhat the relation between material quoted and material presented with proofs, I have referred to every result that either is an exercise or that relies on a treatment outside this book as a proposition, and every result proven in the book as a theorem. This is admittedly quite artificial and obviously gives rise to some strange effects, in that the titles *proposition* and *theorem* are often (if not usually) used to indicate the relative significance of the results discussed. That is not the case here.

There are bibliographic notes at the end of each chapter. They are intended to give the reader some guidance to the background material, and to give but an introduction to a definitive study of the literature.

It is a pleasure to thank the many people with whom I have been associated in the study of geometry since I first came to the City College of CUNY in 1970: first and foremost, Edgar A. Feldman and the other geometers of the City University of New York – J. Dodziuk, L. Karp, B. Randol, R. Sacksteder, and J. Velling. Also, I have benefited through the years from the friendship and mathematics of I. Benjamini, M. van den Berg, P. Buser, E. B. Davies, J. Eels, D. Elworthy, A. A. Grigor'yan, E. Hsu, W. S. Kendall, F. Morgan, R. Osserman, M. Pinsky and D. Sullivan. But, as is well known, any mistakes herein are all mine.

The isoperimetric problem has been a source of mathematical ideas and techniques since its formulation in classical antiquity, and it is still alive and well in its ability to both capture and nourish the mathematical imagination. This book only covers a small portion of the subject; nonetheless, I hope the presentation gives expression to some of its beauty and inspiration.