I

Introduction

In this chapter we introduce the subject. We describe the classical isoperimetric problem in Euclidean space of all dimensions, and give some elementary arguments that work in the plane. Only one approach will carry over to higher dimensions, namely, the necessary condition established by classical calculus of variations, that a domain with C^2 boundary provides a solution to the isoperimetric problem only if it is a disk. Then we give a recent proof of the isoperimetric inequality in the plane by P. Topping (which does not include a characterization of equality), and the classical argument of A. Hurwitz to prove the isoperimetric inequality using Fourier series. This is followed by a symmetry and convexity argument in the plane for very general boundaries that proves the isoperimetric inequality, *if* one assumes in advance that the *isoperimetric functional* $D \mapsto L^2(\partial D)/A(D)$ has a minimizer. (So this is a weak version – if the isoperimetric problem has a solution, then the disk is also a solution.) Finally, we present the background necessary for what follows later in our general discussion, valid for all dimensions. The subsections of §I.3 include a proof of H. Rademacher's theorem on the almost everywhere differentiability of Lipschitz functions, and a proof of the general co-area formula for C^1 mappings of Riemannian manifolds. We obtain the usual co-area formula, as well as an easy consequence: Cauchy's formula for the area of the boundary of a convex subset of $\mathbb{R}^n$ with C^1 boundary.

I.1 The Isoperimetric Problem

Given any bounded domain on the real line (that is, an open interval), the discrete measure of its boundary (the endpoints of the interval) is 2. And given any bounded open subset of the line, the discrete measure of its boundary is greater than or equal to 2, with equality if and only if the open set consists of one open interval. This is the statement of the isoperimetric inequality on the line.

In the plane, one has three common formulations of the *isoperimetric problem*:

1. Consider all bounded domains in $\mathbb{R}^2$ with fixed given perimeter, length of the boundary (that is, all domains under consideration are *isoperimetric*). Find the domain that contains the greatest area. The answer, of course, will be the disk. Note that the specific value of the perimeter in question is of no interest, because all domains of perimeter L_1 are mapped by a similarity of $\mathbb{R}^2$ to all domains with perimeter L_2 for any given values of L_1, L_2, and the image under the similarity of an area maximizer for L_1 is an area maximizer for L_2.
2. One insists on a common area of all bounded domains under consideration, and asks how to minimize the perimeter.
3. Lastly, one expresses the problem as an analytic inequality, namely, since we know exactly the values of the area of the disk and the length of its boundary, the isoperimetric problem is then expressed as proving the *isoperimetric inequality*

   $$L^2 \geq 4\pi A, \tag{I.1.1}$$

   where A denotes the area of the domain under consideration, and L denotes the length of its boundary. The inequality is extremely convenient, in that it remains invariant under similarities of $\mathbb{R}^2$, and one has equality if the domain is a disk. One wishes to show that the inequality is always true, with equality if and only if the domain is a disk.

One can consider the above for any $\mathbb{R}^n$, $n \geq 2$. The proposed analytic isoperimetric inequality then becomes

$$\frac{A(\partial\Omega)}{V(\Omega)^{1-1/n}} \geq \frac{A(\mathbb{S}^{n-1})}{V(\mathbb{B}^n)^{1-1/n}}, \tag{I.1.2}$$

where Ω is any bounded domain in $\mathbb{R}^n$ and $\partial\Omega$ its boundary, V denotes n-measure and A denotes $(n-1)$-measure, $\mathbb{B}^n$ is the unit disk in $\mathbb{R}^n$, and $\mathbb{S}^{n-1}$ the unit sphere in $\mathbb{R}^n$. We let $\boldsymbol{\omega}_\mathbf{n}$ denote the n-dimensional volume of $\mathbb{B}^n$ and $\mathbf{c}_{\mathbf{n-1}}$ the $(n-1)$-dimensional surface area of $\mathbb{S}^{n-1}$. It is standard that

$$\mathbf{c}_{\mathbf{n-1}} = \frac{2\pi^{n/2}}{\Gamma(n/2)}, \qquad \boldsymbol{\omega}_\mathbf{n} = \frac{\mathbf{c}_{\mathbf{n-1}}}{n}, \tag{I.1.3}$$

where $\Gamma(x)$ denotes the classical gamma function; and (I.1.2) now reads as

$$\frac{A(\partial\Omega)}{V(\Omega)^{1-1/n}} \geq n{\boldsymbol{\omega}_\mathbf{n}}^{1/n}. \tag{I.1.4}$$

One wants to prove the inequality and to show that equality is achieved if and only if Ω is an n-disk. Note that for $n = 2$ we took in (I.1.4) the square root of (I.1.1).

Remark I.1.1 Throughout the book, *domain* will refer to a connected open set. In general, we consider the isoperimetric problem for relatively compact domains when we are working in the differential geometric setting (Chapters I, II, V–VIII). Therefore, the disks that realize the solution in $\mathbb{R}^n$ are open. In Chapters III and IV, where we work in a more general setting, the isoperimetric problem is considered for compacta. In that setting the disks that realize the solution in $\mathbb{R}^n$ are closed.

Remark I.1.2 We have restricted the isoperimetric problem to domains in $\mathbb{R}^n$; but if we could solve this problem, then the isoperimetric problem for open sets consisting of finitely many bounded domains would easily follow from the solution for single domains. Indeed, assume one has the inequality (I.1.2) for domains in $\mathbb{R}^n$. If

$$\Omega = \Omega_1 \cup \Omega_2 \cup \cdots,$$

where each Ω_j is a relatively compact domain in $\mathbb{R}^n$ such that

$$\operatorname{cl}\Omega_j \cap \operatorname{cl}\Omega_k = \emptyset \qquad \forall\ j \neq k$$

(cl denotes the closure), then Minkowski's inequality implies

$$\begin{aligned} V(\Omega)^{1-1/n} &\leq \sum_j V(\Omega_j)^{1-1/n} \leq \frac{1}{n{\boldsymbol\omega_{\mathbf n}}^{1/n}} \sum_j A(\partial\Omega_j) \\ &= \frac{1}{n{\boldsymbol\omega_{\mathbf n}}^{1/n}} A(\partial\Omega). \end{aligned} \tag{I.1.5}$$

So the inequality extends to the union of domains. Note that equality implies that Ω is a domain.

Remark I.1.3 Note that for any domain Ω in $\mathbb{R}^n$, its volume is the n-dimensional Lebesgue measure, and if $\partial\Omega$ is C^1 then the area of $\partial\Omega$ is given by the standard differential geometric surface area of a smooth hypersurface in $\mathbb{R}^n$. However, if $\partial\Omega$ is not smooth, then one must propose an area functional defined on a collection of domains such that the area functional will give a working definition of the area of the boundaries of the domains. Besides a number of natural properties [see the discussions in Burago and Zalgaller (1988)], one requires that the new definition agree with the differential geometric one when applied to a domain with smooth boundary. Then, with this new collection of domains and definition of the area of their boundaries, one wishes to prove the isoperimeric inequality. Also, one wishes to characterize the case of equality in each of these settings.

Remark I.1.4 As soon as one expands the problem to the model spaces of constant sectional curvature, that is, to spheres and hyperbolic spaces, one has no self-similarities of the Riemannian spaces in question. And if the disks on the right hand side of (I.1.2) are to have radius r, then the right hand side of the inequality in (I.1.2) is no longer independent of the value of r. Nonetheless, one still has the isoperimetric inequality in the sense that all domains in question with the same n-volume have the $(n-1)$-area of their boundaries minimized by disks. For $n = 2$, the analytic formulation reads as follows: If $M = \mathbb{M}^2_\kappa$, the model space with constant curvature κ, then the isoperimetric inequality becomes

$$L^2 \geq 4\pi A - \kappa A^2, \tag{I.1.6}$$

with equality if and only if the domain in question is a disk. Of course, one can still consider the isoperimetric problem, whether or not it is to be expressed as an inequality, in the first or second formulation above.

Similarly, one can extend the isoperimetric problem and associated inequalities to surfaces, or, more generally, to Riemannian manifolds. We shall consider such inequalities in Chapter V.

Remark I.1.5 Finally, one can consider a *Bonnesen inequality*. In $\mathbb{R}^2$, such an inequality is of the form

$$L^2 - 4\pi A \geq B \geq 0,$$

where B is a nonnegative geometric quantity associated with the domain that vanishes if and only if the domain is a disk.

I.2 The Isoperimetric Inequality in the Plane

For any C^2 path $\omega : (\alpha, \beta) \to \mathbb{R}^2$ in the plane, the velocity vector field of ω is given by its derivative ω', and acceleration vector field by ω''. We assume that ω is an immersion, that is, ω' never vanishes. The infinitesimal element of arc length ds is given by

$$ds = |\omega'(t)|\, dt.$$

Given any $t_0 \in (\alpha, \beta)$, the *arc length function* of ω based at t_0 is given by

$$s(t) = \int_{t_0}^{t} |\omega'(\tau)|\, d\tau.$$

Let

$$\mathbf{T}(t) = \frac{\omega'(t)}{|\omega'(t)|}$$

denote the unit tangent vector field along ω,

$$\iota : \mathbb{R}^2 \to \mathbb{R}^2$$

the rotation of $\mathbb{R}^2$ by $\pi/2$ radians, and

$$\mathbf{N} = \iota\mathbf{T}$$

the oriented unit normal vector field along ω. Then one defines the *curvature* κ of ω by

(I.2.1) $$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$$

(indeed, since $\mathbf{T}$ is a unit vector field, its derivative must be perpendicular to itself). Then the formula for the curvature, relative to the original path, is given by

$$\kappa = \frac{d\mathbf{T}}{ds}\cdot\mathbf{N} = \frac{\omega''\cdot\iota\omega'}{|\omega'|^3}.$$

One can easily show that

(I.2.2) $$\frac{d\mathbf{N}}{ds} = -\kappa\mathbf{T}.$$

The equations (I.2.1) and (I.2.2) are referred to as the *Frenet formulae.*

One can prove, from (I.2.1), that if the curvature κ is constant, then ω is an arc on a circle (if not the complete circle).

I.2.1 Uniqueness for Smooth Boundaries

As a warm-up, we give the argument from classical calculus of variations. Given the area A, let D vary over relatively compact domains in the plane of area A, with C^1 boundary, and suppose the domain Ω, $\partial\Omega \in C^2$, realizes the minimal boundary length among all such domains D. We claim that Ω is a disk.

Proof Since Ω is relatively compact in $\mathbb{R}^2$, there exists a simply connected domain Ω_0 such that

$$\Omega = \Omega_0 \setminus \{\text{finite disjoint union of closed topological disks}\}.$$

We claim that since Ω is a minimizer, then $\Omega_0 = \Omega$. If not, we may add the topological disks to Ω, which will increase the area of the domain and decrease the length of the boundary, and therefore Ω will not be a minimizer. Thus $\Omega_0 = \Omega$, and is bounded by an imbedded circle.

Let $\Gamma : \mathbb{S}^1 \to \mathbb{R}^2 \in C^2$ be the imbedding of the boundary of Ω. We always assume that the path Γ is oriented so that $\boldsymbol{\nu} = -\mathbf{N}$ at all points of Γ, where $\boldsymbol{\nu}$ is the unit normal exterior vector field along $\partial\Omega$.

One then considers a 1-parameter family $\Gamma_\epsilon : \mathbb{S}^1 \to \mathbb{R}^2$ of imbeddings

$$v : (-\epsilon_0, \epsilon_0) \times \mathbb{S}^1 \to \mathbb{R}^2,$$

such that the variation function $v(\epsilon, t)$ given by

$$v(\epsilon, t) = \Gamma_\epsilon(t) = \Gamma(t) + \Psi(\epsilon, t)\boldsymbol{\nu}(t), \qquad \Psi(0, t) = 0,$$

is C^1. First,

$$\frac{\partial v}{\partial \epsilon} = \frac{\partial \Psi}{\partial \epsilon}\boldsymbol{\nu}.$$

Also

$$\frac{\partial v}{\partial t} = \Gamma' + \left\{\frac{\partial \Psi}{\partial t}\boldsymbol{\nu} + \Psi\boldsymbol{\nu}'\right\} = \{1 + \kappa\Psi\}\,\Gamma' + \frac{\partial \Psi}{\partial t}\boldsymbol{\nu},$$

which implies

$$\left|\frac{\partial v}{\partial t}\right| = \left\{(1 + \kappa\Psi)^2 + \frac{1}{|\Gamma'|^2}\left(\frac{\partial \Psi}{\partial t}\right)^2\right\}^{1/2} |\Gamma'|.$$

Taylor's theorem implies, for

$$\phi(t) := \left.\frac{\partial \Psi}{\partial \epsilon}\right|_{\epsilon=0},$$

the expansion

$$\Psi(\epsilon, t) = \epsilon\phi(t) + o(\epsilon), \qquad \frac{\partial \Psi}{\partial \epsilon} = \phi(t) + o(1), \qquad \frac{\partial \Psi}{\partial t} = O(\epsilon),$$

which implies

$$\left|\frac{\partial v}{\partial t}\right| = |\Gamma'|\,\{1 + \epsilon\kappa\phi + o(\epsilon)\}\,.$$

Therefore, the area element dA in the curvilinear coordinates (t, ϵ) is given by

$$dA = \left|\frac{\partial v}{\partial \epsilon} \times \frac{\partial v}{\partial t}\right| d\epsilon\, dt = \phi|\Gamma'|\,\{1 + o(1)\}\, d\epsilon\, dt = \{\phi + o(1)\}\, d\epsilon\, ds.$$

For the domain Ω_ϵ determined by Γ_ϵ we have, for sufficiently small ϵ,

$$A(\Omega_\epsilon) - A(\Omega) = \int_0^\epsilon d\sigma \int_\Gamma \{\phi + o(1)\}\, ds.$$

Therefore, if $A(\Omega_\epsilon) = A(\Omega)$ for all ϵ, then

$$\int_\Gamma \phi\, ds = 0.$$

Now let $L(\epsilon)$ denote the length of Γ_ϵ. Since Γ has the shortest length, we have $L'(0) = 0$. Therefore, because

$$L(\epsilon) = \int_{\mathbb{S}^1} \left|\frac{\partial v}{\partial t}\right| dt = \int_{\mathbb{S}^1} |\Gamma'|\,\{1 + \epsilon\kappa\phi + o(\epsilon)\}\, dt = \int_\Gamma \{1 + \epsilon\kappa\phi + o(\epsilon)\}\, ds,$$

we have

$$0 = L'(0) = \int_\Gamma \kappa\phi\, ds, \qquad \int_\Gamma \phi\, ds = 0$$

for any such variation of Γ.

Similarly, given any $\phi \in C^1$ such that $\int_\Gamma \phi\, ds = 0$, there exists a variation v of Γ such that $A(\Omega_\epsilon) = A(\Omega)$ for all ϵ, and $L'(0) = \int_\Gamma \kappa\phi\, ds$. Then, by assumption, we have

$$\int_\Gamma \kappa\phi\, ds = 0 \qquad \forall \phi \in C^1 : \int_\Gamma \phi\, ds = 0.$$

To show that this implies that κ is constant, we argue as follows: Given *any* $\psi : \mathbb{S}^1 \to \mathbb{R}$ in C^1, set

$$\phi = \psi - \int_\Gamma \psi\, ds \Big/ \int_\Gamma ds\ .$$

Then $\int_\Gamma \phi\, ds = 0$, which implies

$$0 = \int_\Gamma \kappa \left(\psi - \frac{1}{L}\int_\Gamma \psi\, ds\right) ds = \int_\Gamma \left(\kappa - \frac{1}{L}\int_\Gamma \kappa\, ds\right) \psi\, ds,$$

where L denotes the length of Γ. Since ψ is arbitrary C^1, a standard argument then implies that

(I.2.3) $$\kappa - \frac{1}{L}\int_\Gamma \kappa\, ds = 0,$$

that is, the curvature κ is constant. Then, as mentioned, (I.2.1) implies that Γ is a circle.

I.2.2 Quick Proof Using Complex Variables

Theorem I.2.1 (Isoperimetric Inequality in $\mathbb{R}^2$) *Let Ω be a relatively compact domain, with boundary $\partial\Omega \in C^1$ consisting of one component. Then*

$$L^2(\partial\Omega) \geq 4\pi A(\Omega).$$

Proof We denote any element of the plane as the complex number

$$z = x + iy,$$

and the area measure as an oriented volume element; so

$$dA = dx \wedge dy = \frac{i}{2} dz \wedge d\bar{z}.$$

Then

$$\begin{aligned} 4\pi A(\Omega) &= \iint_{\Omega} 2\pi i \, dz \wedge d\bar{z} \\ &= \iint_{\Omega} dz \wedge d\bar{z} \int_{\partial\Omega} \frac{d\zeta}{\zeta - z} \\ &= \int_{\partial\Omega} d\zeta \iint_{\Omega} \frac{dz \wedge d\bar{z}}{\zeta - z} \\ &= \int_{\partial\Omega} d\zeta \int_{\partial\Omega} \frac{\bar{\zeta} - \bar{z}}{\zeta - z} dz \\ &\leq L^2(\partial\Omega), \end{aligned}$$

– the second equality follows from the fact that the winding number of $\partial\Omega$ about any point $z \in \Omega$ is 1; the last equality follows from Green's theorem – which implies the claim. ■

I.2.3 The Method of Fourier Series

Lemma I.2.1 (Wirtinger's Inequality) *If f is a C^1, L-periodic function on $\mathbb{R}$, and*

$$\int_0^L f(t)\, dt = 0,$$

then

$$\int_0^L |f'|^2(t)\, dt \geq \frac{4\pi^2}{L^2} \int_0^L |f|^2(t)\, dt,$$

with equality if and only if there exist constants a_{-1} and a_1 such that

$$f(t) = a_{-1} e^{-2\pi i t/L} + a_1 e^{2\pi i t/L}.$$

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Proof This is an exercise from Fourier series. The function $f(t)$ admits a Fourier expansion

$$f(t) \leftrightarrow \sum_{k=-\infty}^{\infty} a_k e^{2\pi ikt/L} \quad \text{with} \quad a_k = \frac{1}{L}\int_0^L f(t)e^{-2\pi ikt/L}\,dt.$$

Similarly, we have

$$f'(t) \leftrightarrow \sum_{k=-\infty}^{\infty} b_k e^{2\pi ikt/L} \quad \text{with} \quad b_k = \frac{1}{L}\int_0^L f'(t)e^{-2\pi ikt/L}\,dt.$$

The continuity of f implies $b_0 = 0$, and the hypothesis implies $a_0 = 0$. Integration by parts implies

$$b_k = \frac{2\pi ik}{L} a_k \quad \forall\ |k| \geq 1.$$

Parseval's inequality then implies

$$\begin{aligned}\int_0^L |f'|^2\,dt = L\sum_{k\neq 0} |b_k|^2 &= L\frac{4\pi^2}{L^2}\sum_{k\neq 0} k^2|a_k|^2 \\ &\geq L\frac{4\pi^2}{L^2}\sum_{k\neq 0}|a_k|^2 \\ &= \frac{4\pi^2}{L^2}\int_0^L |f|^2\,dt,\end{aligned}$$

which implies the inequality. One has equality if and only if $a_k = 0$ for all $|k| > 1$. ∎

Theorem I.2.2 (Isoperimetric Inequality in $\mathbb{R}^2$) *If Ω is a relatively compact domain in $\mathbb{R}^2$, with C^1 boundary consisting of one component, then*

$$L^2(\partial\Omega) \geq 4\pi A(\Omega),$$

with equality if and only if Ω is a disk.

Proof If necessary, we translate Ω to guarantee

$$\int_{\partial\Omega} x\,ds = 0, \qquad x = (x^1, x^2).$$

Let $\mathbf{x} = x^1\mathbf{e}_1 + x^2\mathbf{e}_2$ be the vector field on $\mathbb{R}^2$ with base point $x = (x^1, x^2)$.

One now uses the 2-dimensional divergence theorem, namely, for any vector field $x \mapsto \boldsymbol{\xi}(x) \in \mathbb{R}^2$ with support containing cl Ω, one has

$$\iint_\Omega \operatorname{div}\boldsymbol{\xi}\,dA = \int_{\partial\Omega} \boldsymbol{\xi}\cdot\boldsymbol{\nu}\,ds, \tag{I.2.4}$$

where $\boldsymbol{\nu}$ denotes the outward unit normal vector field along $\partial\Omega$. One can obtain the formula (I.2.4) by converting the traditional Green's theorem

$$\iint_{\Omega} \left\{ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\} dA = \int_{\partial\Omega} P\,dx + Q\,dy, \tag{I.2.5}$$

by choosing

$$P = -\xi^2, \quad Q = \xi^1, \qquad \boldsymbol{\xi} = \xi^1 \mathbf{e}_1 + \xi^2 \mathbf{e}_2.$$

For the left hand side of (I.2.5) one has

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial \xi^1}{\partial x^1} + \frac{\partial \xi^2}{\partial x^2} = \operatorname{div} \boldsymbol{\xi}.$$

For the right hand side of (I.2.5) one has

$$\begin{aligned} P\,dx + Q\,dy &= -\xi^2\,dx^1 + \xi^1\,dx^2 = \boldsymbol{\xi}\cdot\{dx^2\mathbf{e}_1 - dx^1\mathbf{e}_2\} \\ &= \boldsymbol{\xi}\cdot\{-\iota(d\mathbf{x})\} = \boldsymbol{\xi}\cdot\boldsymbol{\nu}\,ds. \end{aligned}$$

For our vector field $\mathbf{x}$, we have $\operatorname{div}\mathbf{x} = 2$ on all Ω. Then the divergence theorem implies

$$2A(\Omega) = \int_{\Omega} \operatorname{div}\mathbf{x}\,dA = \int_{\partial\Omega} \mathbf{x}\cdot\boldsymbol{\nu}\,ds,$$

which implies

$$\begin{aligned} 2A(\Omega) &= \int_{\partial\Omega} \mathbf{x}\cdot\boldsymbol{\nu}\,ds \\ &\le \int_{\partial\Omega} |\mathbf{x}|\,ds \\ &\le \left\{ \int_{\partial\Omega} |\mathbf{x}|^2\,ds \right\}^{1/2} \left\{ \int_{\partial\Omega} 1^2\,ds \right\}^{1/2} \\ &= L^{1/2}(\partial\Omega) \left\{ \int_{\partial\Omega} |\mathbf{x}|^2\,ds \right\}^{1/2} \end{aligned}$$

– the first inequality is the vector Cauchy–Schwarz inequality, and the second inequality is the integral Cauchy–Schwarz inequality.

Parametrize $\partial\Omega$ with respect to arc length. Note that

$$|\mathbf{x}|^2 = (x^1)^2 + (x^2)^2, \qquad \left| \frac{d\mathbf{x}}{ds} \right|^2 = \left(\frac{dx^1}{ds} \right)^2 + \left(\frac{dx^2}{ds} \right)^2$$