141 Fixed Point Theory and Applications

This book provides a clear exposition of the flourishing field of fixed point theory. Starting from the basics of Banach's contraction theorem, most of the main results and techniques are developed: fixed point results are established for several classes of maps and the three main approaches to establishing continuation principles are presented. The theory is applied to many areas of current interest in analysis. Topological considerations play a crucial role, including a final chapter on the relationship with degree theory. Researchers and graduate students in applicable analysis will find this to be a useful survey of the fundamental principles of the subject. The very extensive bibliography and close to 100 exercises mean that it can be used both as a text and as a comprehensive reference work, currently the only one of its type.

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Preface

Perhaps the most well known result in the theory of fixed points is Banach's contraction mapping principle. It is therefore fitting that we commence this book with a discussion of contractions and a proof of this result. In addition in Chapter 1, a local version and a generalisation of Banach's contraction theorem are presented. We choose the problem of existence and uniqueness of solutions of certain first order initial value problems to demonstrate the results detailed in the chapter.

It is inevitable that any discussion on contractive maps will lead naturally to another on nonexpansive maps, which is why we choose this as the topic of Chapter 2. Schauder's theorem for nonexpansive maps is presented but the main theorem discussed is a result proved independently in 1965 by Browder, Göhde and Kirk which shows that each nonexpansive map $F : C \to C$, where C is a particular set in a Hilbert space, has at least one fixed point. As a natural lead in to the next chapter, we close Chapter 2 with a nonlinear alternative of Leray–Schauder type for nonexpansive maps.

Chapter 3 is concerned with continuation methods for contractive and nonexpansive maps. We show initially that the property of having a fixed point is invariant by homotopy for contractions. Using this result a nonlinear alternative of Leray–Schauder type is presented for contractive maps and subsequently generalised for nonexpansive maps. An application of the nonlinear alternative for contractions is demonstrated with a second order homogeneous Dirichlet problem.

Fixed point theory for continuous, single valued maps in finite and infinite dimensional Banach spaces is discussed in Chapter 4 with the theorems of Brouwer, Schauder and Mönch presented. The first half of the chapter is devoted to proving and generalising the result of Brouwer which states that every continuous map $F : B^n \to B^n$, where B^n is the closed unit ball in $\mathbf{R}^n$, has a fixed point. An extension of Brouwer's theorem to compact maps in normed linear spaces is presented in the well known fixed point theorem of Schauder, the proof of which relies on Schauder projections. A further generalisation of this theorem due to Mönch in 1980 is also presented. Applications of theorems in Chapter 4 are illustrated with a discrete boundary value problem.

The problem with the results presented in Chapter 4 – from an applications viewpoint – is that they require the map under investigation to take a closed convex set back into itself. As a result, in Chapter 5 we turn our attention to establishing a fixed point theory for nonself maps. However, Schauder's fixed point theorem from Chapter 4 enables us to obtain a nonlinear alternative for continuous, compact (nonself) maps, which we immediately apply to establish an existence principle for a nonlinear Fredholm integral equation. Two existence results are subsequently presented, each ensuring the existence of at least one continuous solution of the equation. In a similar fashion, using Mönch's fixed point theorem from Chapter 4, we obtain a nonlinear alternative for continuous, Mönch type maps which further leads to a variety of nonlinear alternatives for condensing maps, k-set contractive maps, and maps of the form $F := F_1 + F_2$ where F_1 and F_2 satisfy certain conditions. The chapter closes by discussing maps $F : X \to E$ where (unlike the other theorems presented in the chapter) the interior of X may be empty.

Having discussed condensing maps in Chapter 4 it is now natural to consider continuation principles for these maps. There are three main approaches in the literature. The first approach uses degree theory (Chapter 12), the second is the essential map approach of Granas and the third is the 0-epi map approach of Furi, Martelli and Vignoli (Chapter 8). In Chapter 6 we discuss the second approach. The chapter is devoted to showing that the property of having a fixed point (or more generally, being essential) is invariant by homotopy for compact (or more generally, condensing) maps. Using the theory of essential maps, a nonlinear alternative is presented firstly for continuous, compact maps, with an analogous result for continuous, condensing maps appearing towards the end of the chapter. In addition, using results obtained for the completely continuous field, $f(x) = x - F(x)$, the equation $y = x - F(x)$ is discussed, with the Fredholm alternative making a guest appearance as an immediate consequence.

Fixed point theorems in conical shells are discussed in Chapter 7. We consider continuous, compact maps $F : B \to C$, where C is a cone. Specifically we present Krasnoselskii's compression and expansion of a cone theorems, the proofs of which rely on the essential map theory of the previous chapter. Krasnoselskii's theorems are of particular importance when it comes to establishing the existence of multiple solutions of operator equations. A large portion of the chapter is therefore assigned to using these results to prove the existence of multiple solutions of a nonlinear Fredholm integral equation, similar to that discussed already in Chapter 5. In fact, in certain instances we combine the nonlinear alternative for compact maps presented in Chapter 5 and the fixed point theorems of this chapter, to obtain stronger results.

In Chapter 8 we present fixed point results for maps defined on Hausdorff locally convex linear topological spaces. The extension of Schauder's fixed point theorem to such spaces is known as the Schauder–Tychonoff theorem and this is the first main result of the chapter. With this established we then present a nonlinear alternative of Leray–Schauder type for maps defined on the spaces in question. This in turn is used to prove a fixed point theorem of Furi–Pera type which we require to obtain existence principles and results for two types of second order boundary value problems on the half-line. The chapter is concluded with a continuation principle for continuous, compact maps defined on Fréchet spaces. In particular, the 0-epi map approach of Furi, Martelli and Vignoli is presented.

Contractive and nonexpansive multivalued mappings are discussed in Chapter 9. We first concentrate on contractive multivalued mappings. The result due to Nadler of the Banach contraction principle for contractive mappings with closed values is first presented. Once we have shown that the property of having a fixed point is invariant by homotopy for contractive multivalued mappings, a nonlinear alternative of Leray–Schauder type for such maps is then discussed. Finally we extend this result to nonexpansive multivalued mappings.

Chapter 10 deals with multivalued maps which have continuous selection. For the convenience of the reader we restrict our attention to one particular class of maps, namely the $\Phi^\star$ maps. A nonlinear alternative of Leray–Schauder type, a Furi–Pera type result and some coincidence type results are just some of the fixed point theory presented for $\Phi^\star$ maps. An application to abstract economies concludes the discussion.

The objective of Chapter 11 is to extend the Schauder–Tychonoff theorem to multivalued maps with closed graph. The basic results in this chapter are due to S. Kakutani, I. L. Glicksberg and K. Fan. Much of the chapter is spent working towards the proof of Ky Fan's minimax theorem which is the crucial result needed to prove the analogue of the

Schauder–Tychonoff theorem for these maps. The chapter closes with a nonlinear alternative for multivalued maps with closed graph.

A chapter on degree theory concludes the book. The concept of the degree of a map is introduced and is used to provide another approach to presenting fixed point theory and continuation principles. The first half of the chapter deals with the degree of a map defined on subsets of $\mathbf{R}^n$ and a proof of Brouwer's fixed point theorem using the degree theory established is just one of the results presented. The second half of the chapter looks at the degree of a map defined on a normed linear space and here an alternative proof of Schauder's fixed point theorem using degree theory is illustrated.