1

Contractions

Let (X, d) be a metric space. A map $F : X \to X$ is said to be *Lipschitzian* if there exists a constant $\alpha \geq 0$ with

$$d(F(x), F(y)) \leq \alpha \, d(x, y) \text{ for all } x, y \in X. \tag{1.1}$$

Notice that a Lipschitzian map is necessarily continuous. The smallest α for which (1.1) holds is said to be the *Lipschitz constant* for F and is denoted by L. If $L < 1$ we say that F is a *contraction*, whereas if $L = 1$, we say that F is *nonexpansive.*

For notational purposes we define $F^n(x)$, $x \in X$ and $n \in \{0, 1, 2, \ldots\}$, inductively by $F^0(x) = x$ and $F^{n+1}(x) = F(F^n(x))$.

The first result in this chapter is known as Banach's contraction principle.

Theorem 1.1 *Let (X, d) be a complete metric space and let $F : X \to X$ be a contraction with Lipschitzian constant L. Then F has a unique fixed point $u \in X$. Furthermore, for any $x \in X$ we have*

$$\lim_{n \to \infty} F^n(x) = u$$

with

$$d(F^n(x), u) \leq \frac{L^n}{1 - L} \, d(x, F(x)).$$

Proof We first show uniqueness. Suppose there exist $x, y \in X$ with $x = F(x)$ and $y = F(y)$. Then

$$d(x, y) = d(F(x), F(y)) \leq L \, d(x, y),$$

therefore $d(x, y) = 0$.

To show existence select $x \in X$. We first show that $\{F^n(x)\}$ is a Cauchy sequence. Notice for $n \in \{0, 1, \ldots\}$ that

$$d(F^n(x), F^{n+1}(x)) \leq L\, d(F^{n-1}(x), F^n(x)) \leq \cdots \leq L^n\, d(x, F(x)).$$

Thus for $m > n$ where $n \in \{0, 1, \ldots\}$,

$$\begin{aligned} d(F^n(x), F^m(x)) &\leq d(F^n(x), F^{n+1}(x)) + d(F^{n+1}(x), F^{n+2}(x)) \\ &\quad + \cdots + d(F^{m-1}(x), F^m(x)) \\ &\leq L^n\, d(x, F(x)) + \cdots + L^{m-1}\, d(x, F(x)) \\ &\leq L^n\, d(x, F(x)) \left[1 + L + L^2 + \cdots\right] \\ &= \frac{L^n}{1-L}\, d(x, F(x)). \end{aligned}$$

That is for $m > n$, $n \in \{0, 1, \ldots\}$,

$$d(F^n(x), F^m(x)) \leq \frac{L^n}{1-L}\, d(x, F(x)). \tag{1.2}$$

This shows that $\{F^n(x)\}$ is a Cauchy sequence and since X is complete there exists $u \in X$ with $\lim_{n\to\infty} F^n(x) = u$. Moreover the continuity of F yields

$$u = \lim_{n\to\infty} F^{n+1}(x) = \lim_{n\to\infty} F(F^n(x)) = F(u),$$

therefore u is a fixed point of F. Finally letting $m \to \infty$ in (1.2) yields

$$d(F^n(x), u) \leq \frac{L^n}{1-L}\, d(x, F(x)).$$ □

Remark 1.1 Theorem 1.1 requires that $L < 1$. If $L = 1$ then F need not have a fixed point as the example $F(x) = x + 1$ for $x \in \mathbf{R}$ shows. We will discuss the case when $L = 1$ in more detail in Chapter 2.

Another natural attempt to extend Theorem 1.1 would be to suppose that $d(F(x), F(y)) < d(x, y)$ for $x, y \in X$ with $x \neq y$. Again F need not have a fixed point as the example $F(x) = \ln(1 + e^x)$ for $x \in \mathbf{R}$ shows. However there is a positive result along these lines in the following theorem of Edelstein.

Theorem 1.2 *Let (X, d) be a compact metric space with $F : X \to X$ satisfying*

$$d(F(x), F(y)) < d(x, y) \ \textit{for } x,\ y \in X \textit{ and } x \neq y.$$

Then F has a unique fixed point in X.

Proof The uniqueness part is easy. To show existence, notice the map $x \mapsto d(x, F(x))$ attains its minimum, say at $x_0 \in X$. We have $x_0 = F(x_0)$ since otherwise

$$d(F(F(x_0)), F(x_0)) < d(F(x_0), x_0)$$

– a contraction. □

We next present a local version of Banach's contraction principle. This result will be needed in Chapter 3.

Theorem 1.3 *Let (X, d) be a complete metric space and let*

$$B(x_0, r) = \{x \in X : d(x, x_0) < r\}, \text{ where } x_0 \in X \text{ and } r > 0.$$

Suppose $F : B(x_0, r) \to X$ is a contraction (that is, $d(F(x), F(y)) \leq L\, d(x, y)$ for all x, $y \in B(x_0, r)$ with $0 \leq L < 1$) with

$$d(F(x_0), x_0) < (1 - L)\, r.$$

Then F has a unique fixed point in $B(x_0, r)$.

Proof There exists r_0 with $0 \leq r_0 < r$ with $d(F(x_0), x_0) \leq (1 - L)r_0$. We will show that $F : \overline{B(x_0, r_0)} \to \overline{B(x_0, r_0)}$. To see this note that if $x \in \overline{B(x_0, r_0)}$ then

$$\begin{aligned} d(F(x), x_0) &\leq d(F(x), F(x_0)) + d(F(x_0), x_0) \\ &\leq L\, d(x, x_0) + (1 - L)r_0 \leq r_0. \end{aligned}$$

We can now apply Theorem 1.1 to deduce that F has a unique fixed point in $\overline{B(x_0, r_0)} \subset B(x_0, r)$. Again it is easy to see that F has only one fixed point in $B(x_0, r)$. □

Next we examine briefly the behaviour of a contractive map defined on $\overline{B}_r = \overline{B(0, r)}$ (the closed ball of radius r with centre 0) with values in a Banach space E. More general results will be presented in Chapter 3.

Theorem 1.4 *Let $\overline{B}_r$ be the closed ball of radius $r > 0$, centred at zero, in a Banach space E with $F : \overline{B}_r \to E$ a contraction and $F(\partial \overline{B}_r) \subseteq \overline{B}_r$. Then F has a unique fixed point in $\overline{B}_r$.*

Proof Consider

$$G(x) = \frac{x + F(x)}{2}.$$

We first show that $G : \overline{B}_r \to \overline{B}_r$. To see this let

$$x^\star = r\frac{x}{\|x\|} \text{ where } x \in \overline{B}_r \text{ and } x \neq 0.$$

Now if $x \in \overline{B}_r$ and $x \neq 0$,

$$\|F(x) - F(x^\star)\| \le L\,\|x - x^\star\| = L\,(r - \|x\|),$$

since $x - x^\star = \dfrac{x}{\|x\|}(\|x\| - r)$, and as a result

$$\begin{aligned} \|F(x)\| &\le \|F(x^\star)\| + \|F(x) - F(x^\star)\| \\ &\le r + L(r - \|x\|) \le 2r - \|x\|. \end{aligned}$$

Then for $x \in \overline{B}_r$ and $x \neq 0$

$$\|G(x)\| = \left\|\frac{x + F(x)}{2}\right\| \le \frac{\|x\| + \|F(x)\|}{2} \le r.$$

In fact by continuity we also have

$$\|G(0)\| \le r,$$

and consequently $G : \overline{B}_r \to \overline{B}_r$. Moreover $G : \overline{B}_r \to \overline{B}_r$ is a contraction since

$$\|G(x) - G(y)\| \le \frac{\|x - y\| + L\|x - y\|}{2} = \frac{[1 + L]}{2}\|x - y\|.$$

Theorem 1.1 implies that G has a unique fixed point $u \in \overline{B}_r$. Of course if $u = G(u)$ then $u = F(u)$. □

Over the last fifty years or so, many authors have given generalisations of Banach's contraction principle. Here for completeness we give one such result. Its proof relies on the following technical result.

Theorem 1.5 *Let (X, d) be a complete metric space and $F : X \to X$ a map (not necessarily continuous). Suppose the following condition holds:*

$$(1.3) \qquad \begin{cases} \text{for each } \epsilon > 0 \text{ there is a } \delta(\epsilon) > 0 \text{ such that if} \\ d(x, F(x)) < \delta(\epsilon), \text{ then } F(B(x, \epsilon)) \subseteq B(x, \epsilon); \\ \text{here } B(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}. \end{cases}$$

If for some $u \in X$ we have

$$\lim_{n\to\infty} d(F^n(u), F^{n+1}(u)) = 0,$$

then the sequence $\{F^n(u)\}$ converges to a fixed point of F.

Proof Let u be as described above and let $u_n = F^n(u)$. We claim that $\{u_n\}$ is a Cauchy sequence.

Let $\epsilon > 0$ be given. Choose $\delta(\epsilon)$ as in (1.3). We can choose N large enough so that

$$d(u_n, u_{n+1}) < \delta(\epsilon) \text{ for all } n \geq N.$$

Now since $d(u_N, F(u_N)) < \delta(\epsilon)$, then (1.3) guarantees that

$$F\left(B(u_N, \epsilon)\right) \subseteq B(u_N, \epsilon),$$

and so $F(u_N) = u_{N+1} \in B(u_N, \epsilon)$. Now by induction

$$F^k(u_N) = u_{N+k} \in B(u_N, \epsilon) \text{ for all } k \in \{0, 1, 2, \ldots\}.$$

Thus

$$d(u_k, u_l) \leq d(u_k, u_N) + d(u_N, u_l) < 2\epsilon \text{ for all } k, l \geq N,$$

and therefore $\{u_n\}$ is a Cauchy sequence. In addition there exists $y \in X$ with $\lim_{n\to\infty} u_n = y$.

We now claim that y is a fixed point of F. Suppose it is not. Then

$$d(y, F(y)) = \gamma > 0.$$

We can now choose (and fix) a $u_n \in B(y, \gamma/3)$ with

$$d(u_n, u_{n+1}) < \delta(\gamma/3).$$

Now (1.3) guarantees that

$$F\left(B(u_n, \gamma/3)\right) \subseteq B\left(u_n, \gamma/3\right),$$

and consequently $F(y) \in B(u_n, \gamma/3)$. This is a contradiction since

$$d(F(y), u_n) \geq d(F(y), y) - d(u_n, y) > \gamma - \frac{\gamma}{3} = \frac{2\gamma}{3}.$$

Thus $d(y, F(y)) = 0$. □

Theorem 1.6 *Let (X, d) be a complete metric space and let*

$$d(F(x), F(y)) \leq \phi(d(x, y)) \textit{ for all } x, y \in X;$$

here $\phi : [0, \infty) \to [0, \infty)$ is any monotonic, nondecreasing (not necessarily continuous) function with $\lim_{n\to\infty} \phi^n(t) = 0$ for any fixed $t > 0$. Then F has a unique fixed point $u \in X$ with

$$\lim_{n\to\infty} F^n(x) = u \textit{ for each } x \in X.$$

Proof Suppose $t \leq \phi(t)$ for some $t > 0$. Then $\phi(t) \leq \phi(\phi(t))$ and therefore $t \leq \phi^2(t)$. By induction, $t \leq \phi^n(t)$ for $n \in \{1, 2, \dots\}$. This is a contradiction. Thus $\phi(t) < t$ for each $t > 0$.

In addition,

$$d(F^n(x), F^{n+1}(x)) \leq \phi^n(d(x, F(x))) \text{ for } x \in X,$$

and therefore

$$\lim_{n\to\infty} d(F^n(x), F^{n+1}(x)) = 0 \text{ for each } x \in X.$$

Let $\epsilon > 0$ and choose $\delta(\epsilon) = \epsilon - \phi(\epsilon)$. If $d(x, F(x)) < \delta(\epsilon)$, then for any $z \in B(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$ we have

$$\begin{aligned} d(F(z), x) &\leq d(F(z), F(x)) + d(F(x), x) \leq \phi(d(z, x)) + d(F(x), x) \\ &< \phi(d(z, x)) + \delta(\epsilon) \leq \phi(\epsilon) + (\epsilon - \phi(\epsilon)) = \epsilon, \end{aligned}$$

and therefore $F(z) \in B(x, \epsilon)$. Theorem 1.5 guarantees that F has a fixed point u with $\lim_{n\to\infty} F^n(x) = u$ for each $x \in X$. Finally it is easy to see that F has only one fixed point in X. □

Remark 1.2 Note that Theorem 1.1 follows as a special case of Theorem 1.6 if we choose $\phi(t) = Lt$ with $0 \leq L < 1$.

It is natural to begin our applications of fixed point methods with existence and uniqueness of solutions of certain first order initial value problems. In particular we seek solutions to

$$\begin{cases} y'(t) = f(t, y(t)), \\ y(0) = y_0, \end{cases} \tag{1.4}$$

where $f : I \times \mathbf{R}^n \to \mathbf{R}^n$ and $I = [0, b]$. Notice that (1.4) is a system of first order equations because f takes values in $\mathbf{R}^n$.

We begin our analysis of (1.4) by assuming that $f : I \times \mathbf{R}^n \to \mathbf{R}^n$ is continuous. Then, evidently, $y \in C^1(I)$ (the Banach space of functions u whose first derivative is continuous on I and equipped with the norm $|u|_1 = \max\{\sup_{t\in I} |u(t)|, \sup_{t\in I} |u'(t)|\}$) solves (1.4) if and only if $y \in C(I)$ (the Banach space of functions u, continuous on I and equipped with the norm $|u|_0 = \sup_{t\in I} |u(t)|$) solves

$$y(t) = y_0 + \int_0^t f(s, y(s))\, ds. \tag{1.5}$$

Define an integral operator $T : C(I) \to C(I)$ by

$$Ty(t) = y_0 + \int_0^t f(s, y(s))\, ds.$$

Then the equivalence above is expressed briefly by

$$y \text{ solves (1.4) if and only if } y = Ty, \;\; T : C(I) \to C(I).$$

In other words, classical solutions to (1.4) are fixed points of the integral operator T. We now present a result known as the Picard–Lindelöf theorem.

Theorem 1.7 *Let $f : I \times \mathbf{R}^n \to \mathbf{R}^n$ be continuous and Lipschitz in y; that is, there exists $\alpha \geq 0$ such that*

$$|f(t, y) - f(t, z)| \leq \alpha\, |y - z| \text{ for all } y,\, z \in \mathbf{R}^n.$$

Then there exists a unique $y \in C^1(I)$ that solves (1.4).

Proof We will apply Theorem 1.1 to show that T has a unique fixed point. At first glance it seems natural to use the maximum norm on $C(I)$, but this choice would lead us only to a local solution defined on a subinterval of I. The trick is to use the weighted maximum norm

$$\|y\|_\alpha = |e^{-\alpha t} y(t)|_0$$

on $C(I)$. Observe that $C(I)$ is a Banach space with this norm since it is equivalent to the maximum norm, that is,

$$e^{-\alpha b} |y|_0 \leq \|y\|_\alpha \leq |y|_0.$$

We now show that T is a contraction on $(C(I), \|\cdot\|_\alpha)$. To see this let $y, z \in C(I)$ and notice

$$Ty(t) - Tz(t) = \int_0^t [f(s, y(s)) - f(s, z(s))]\, ds \text{ for } t \in I.$$

Thus for $t \in I$,

$$\begin{aligned} e^{-\alpha t} |(Ty - Tz)(t)| &\leq e^{-\alpha t} \int_0^t \alpha e^{\alpha s} e^{-\alpha s} |y(s) - z(s)|\, ds \\ &\leq e^{-\alpha t} \left(\int_0^t \alpha e^{\alpha s}\, ds \right) \|y - z\|_\alpha \\ &\leq e^{-\alpha t} \left(e^{\alpha t} - 1 \right) \|y - z\|_\alpha \\ &\leq (1 - e^{-\alpha b}) \|y - z\|_\alpha, \end{aligned}$$

and therefore

$$\|Ty - Tz\|_\alpha \le \left(1 - e^{-\alpha b}\right) \|y - z\|_\alpha.$$

Since $1 - e^{-\alpha b} < 1$, the Banach contraction principle implies that there is a unique $y \in C(I)$ with $y = Ty$; equivalently (1.4) has a unique solution $y \in C^1(I)$. □

Now we relax the continuity assumption on f and extend the notion of a solution of (1.4) accordingly. We want to do this in a way that preserves the natural equivalence between (1.4) and the equation $y = Ty$, which was obtained by integrating. To this end we follow the ideas of Carathéodory and make the following definitions.

Definition 1.1 A function $y \in W^{1,p}(I)$ is an L^p*-Carathéodory solution* of (1.4) if y solves (1.4) in the almost everywhere sense on I; here $W^{1,p}(I)$ is the Sobolev class of functions u, with u absolutely continuous and $u' \in L^p(I)$.

Definition 1.2 A function $f : I \times \mathbf{R}^n \to \mathbf{R}^n$ is an L^p*-Carathéodory function* if it satisfies the following conditions:

(c1) the map $y \mapsto f(t, y)$ is continuous for almost every $t \in I$;
(c2) the map $t \mapsto f(t, y)$ is measurable for all $y \in \mathbf{R}^n$;
(c3) for every $c > 0$ there exists $h_c \in L^p(I)$ such that $|y| \le c$ implies that $|f(t, y)| \le h_c(t)$ for almost every $t \in I$.

If f is an L^p-Carathéodory function, then $y \in W^{1,p}(I)$ solves (1.4) if and only if

$$y \in C(I) \text{ and } y(t) = y_0 + \int_0^t f(s, y(s))\, ds.$$

In fact (c1) and (c2) imply that the integrand on the right is measurable for any measurable y, and (c3) guarantees that it is integrable for any bounded measurable y. The stated equivalence now is clear. Therefore just as in the continuous case,

$$(1.4) \text{ has a solution } y \text{ if and only if } y = Ty, \;\; T : C(I) \to C(I).$$

Theorem 1.8 *Let $f : I \times \mathbf{R}^n \to \mathbf{R}^n$ be an L^p-Carathéodory function and L^p-Lipschitz in y; that is, there exists $\alpha \in L^p(I)$ with*

$$|f(t, y) - f(t, z)| \le \alpha(t)|y - z| \text{ for all } y,\ z \in \mathbf{R}^n.$$

Then there exists a unique $y \in W^{1,p}(I)$ that solves (1.4).

Proof The proof is similar to Theorem 1.7 and will only be sketched here. Let

$$A(t) = \int_0^t \alpha(s)\, ds.$$

Then $A'(t) = \alpha(t)$ for a.e. t. Define

$$\|y\|_A = \left| e^{-A(t)} y(t) \right|_0 .$$

The norm is equivalent to the maximum norm because

$$e^{-\|\alpha\|_1} |y|_0 \le \|y\|_A \le |y|_0, \text{ where } \|\alpha\|_1 = \int_0^b |\alpha(t)|\, dt.$$

Thus $(C(I), \|\cdot\|_A)$ is a Banach space and use of the Banach contraction principle, essentially as in the proof of Theorem 1.7, implies that there exists a unique $y \in C(I)$ with $y = Ty$. It follows that (1.4) has a unique L^p-Carathéodory solution on I. □

Notes Most of the results in Chapter 1 may be found in the classical books of Dugundji and Granas [55], Goebel and Kirk [77] and Zeidler [191].

Exercises

1.1 Show that a contraction F from an incomplete metric space into itself need not have a fixed point.

1.2 Let (X, d) be a complete metric space and let $F : X \to X$ be such that $F^N : X \to X$ is a contraction for some positive integer N. Show that F has a unique fixed point $u \in X$ and that for each $x \in X$, $\lim_{n\to\infty} F^n(x) = u$.

1.3 Using the result obtained in Exercise 1.2, give an alternative proof for the Picard–Lindelöf theorem (Theorem 1.7).

1.4 Let $\overline{B}_r$ be the closed ball of radius $r > 0$, centred at zero, in a Banach space E with $F : \overline{B}_r \to E$ a contraction and $F(-x) = -F(x)$ for $x \in \partial \overline{B}_r$. Show F has a fixed point in $\overline{B}_r$.

1.5 Let U be an open subset of a Banach space E and let $F : U \to E$ be a contraction. Show that $(I - F)(U)$ is open.

1.6 Let (X,d) be a complete metric space, P a topological space and $F : X \times P \to X$. Suppose F is a contraction uniformly over P (that is, for each $x, y \in X$, $d(F(x,p), F(y,p)) \leq L\, d(x,y)$ for all $p \in P$) and is continuous in p for each fixed $x \in X$. Let x_p be the unique fixed point of $F_p : X \to X$, where $F_p(x) = F(x,p)$. Show that $p \mapsto x_p$ is continuous.

1.7 Let $k : [0,1] \times [0,1] \times \mathbf{R} \to \mathbf{R}$ be continuous with

$$|k(t,s,x) - k(t,s,y)| \leq L\,|x - y|$$

for all $(t,s) \in [0,1] \times [0,1]$ and $x, y \in \mathbf{R}$ (here $L \geq 0$ is a constant) and $v \in C[0,1]$.

(a) Show that

$$u(t) = v(t) + \int_0^t k(t,s,u(s))\,ds, \quad 0 \leq t \leq 1,$$

has a unique solution $u \in C[0,1]$.

(b) Choose $u_0 \in C[0,1]$ and define a sequence of functions $\{u_n\}$ inductively by

$$u_{n+1}(t) = v(t) + \int_0^t k(t,s,u_n(s))\,ds, \quad n = 0, 1, \ldots .$$

Show that the sequence $\{u_n\}$ converges uniformly on $[0,1]$ to the unique solution $u \in C[0,1]$.

1.8 Let (X,d) be a complete metric space and let $\phi : X \to [0,\infty)$ be a map (not necessarily continuous). Suppose

$$\inf\{\phi(x) + \phi(y) : d(x,y) \geq \gamma\} = \mu(\gamma) > 0 \text{ for all } \gamma > 0.$$

Show that each sequence $\{x_n\}$ in X, for which $\lim_{n\to\infty} \phi(x_n) = 0$, converges to one and only one point $u \in X$.

1.9 Let (X,d) be a complete metric space and let $F : X \to X$ be continuous. Suppose $\phi(x) = d(x, F(x))$ satisfies

$$\inf\{\phi(x) + \phi(y) : d(x,y) \geq \gamma\} = \mu(\gamma) > 0 \text{ for all } \gamma > 0,$$

and that $\inf_{x \in X} d(x, F(x)) = 0$. Show that F has a unique fixed point.

1.10 If in Theorem 1.6 the assumptions on ϕ are replaced by $\phi : [0,\infty) \to [0,\infty)$ is upper semicontinuous from the right on $[0,\infty)$ (that is, $\limsup_{s \to t^+} \phi(s) \leq \phi(t)$ for $t \in [0,\infty)$) and satisfies