Topics in Topological Graph Theory

The use of topological ideas to explore various aspects of graph theory, and vice versa, is a fruitful area of research. There are links with other areas of mathematics, such as design theory and geometry, and increasingly with such areas as computer networks where symmetry is an important feature. Other books cover portions of the material here, but there are no other books with such a wide scope.

This book contains fifteen expository chapters written by acknowledged international experts in the field. Their well-written contributions have been carefully edited to enhance readability and to standardize the chapter structure, terminology and notation throughout the book. To help the reader, there is an extensive introductory chapter that covers the basic background material in graph theory and the topology of surfaces. Each chapter concludes with an extensive list of references.

LOWELL W. BEINEKE is Schrey Professor of Mathematics at Indiana University–Purdue University Fort Wayne, where he has been since receiving his Ph.D. from the University of Michigan under the guidance of Frank Harary. His graph theory interests are broad, and include topological graph theory, line graphs, tournaments, decompositions and vulnerability. With Robin Wilson he edited *Selected Topics in Graph Theory* (3 volumes), *Applications of Graph Theory, Graph Connections* and *Topics in Algebraic Graph Theory*. Until recently he was editor of the *College Mathematics Journal*.

ROBIN J. WILSON is Professor of Pure Mathematics at The Open University, UK, and Emeritus Professor of Geometry at Gresham College, London. After graduating from Oxford, he received his Ph.D. in number theory from the University of Pennsylvania. He has written and edited many books on graph theory and the history of mathematics, including *Introduction to Graph Theory* and *Four Colours Suffice*, and his research interests include graph colourings and the history of combinatorics. He has won a Lester Ford Award and a George Pólya Award from the MAA for his expository writing.

JONATHAN L. GROSS, Professor of Computer Science at Columbia University, served as an academic consultant for this volume. His mathematical work in topology and graph theory have earned him an Alfred P. Sloan Fellowship, an IBM Postdoctoral Fellowship, and numerous research grants. With Thomas Tucker, he wrote *Topological Graph Theory* and several fundamental pioneering papers on voltage graphs and on enumerative methods. He has written and edited eight books on graph theory and combinatorics, seven books on computer programming topics, and one book on cultural sociometry.

THOMAS W. TUCKER, Charles Hetherington Professor of Mathematics at Colgate University, also served as an academic consultant for this volume. He has been at Colgate University since 1973, after a Ph.D. in 3-manifolds from Dartmouth in 1971 and a post-doctoral position at Princeton. He is co-author (with Jonathan Gross) of *Topological Graph Theory.* His early publications were on non-compact 3-manifolds, then topological graph theory, but his recent work is mostly algebraic, especially distinguishability and the group-theoretic structure of symmetric maps.

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Leonhard Euler (1707–1783),
the founder of topological graph theory.

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Edited by

LOWELL W. BEINEKE
Indiana University–Purdue University
Fort Wayne

ROBIN J. WILSON
The Open University

Academic Consultants

JONATHAN L. GROSS
Columbia University

THOMAS W. TUCKER
Colgate University

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This book is dedicated to the memory of Gerhard Ringel (1919–2008),
one of the pioneers of modern topological graph theory.

Contents

Foreword

The origins of topological graph theory lie in the 19th century, largely with the four colour problem and its extension to higher-order surfaces – the Heawood map problem. With the explosive growth of topology in the early 20th century, mathematicians like Veblen, Rado and Papakyriakopoulos provided foundational results for understanding surfaces combinatorially and algebraically. Kuratowski, MacLane and Whitney in the 1930s approached the four colour problem as a question about the structure of graphs that can be drawn without edge-crossings in the plane. Kuratowski's theorem characterizing planarity by two obstructions is the most famous, and its generalization to the higher-order surfaces became an influential unsolved problem.

The second half of the 20th century saw the solutions of all three problems: the Heawood map problem by Ringel, Youngs *et al.* by 1968, the four colour problem by Appel and Haken in 1976, and finally the generalized Kuratowski problem by Robertson and Seymour in the mid-1990s. Each is a landmark of 20th-century mathematics. The Ringel–Youngs work led to an alliance between combinatorics and the algebraic topology of branched coverings. The Appel–Haken work was the first time that a mathematical theorem relied on exhaustive computer calculations. And the Robertson–Seymour work led to their solution of Wagner's conjecture, which provides a breathtaking structure for the collection of all finite graphs, a collection that would seem to have no structure at all.

Each of these problems centres on the question of which graphs can be embedded in which surfaces, with two complementary perspectives – fixing the graph or fixing the surface. Although the question sounds highly focused, the study of graphs on surfaces turns out to be incredibly broad, rich in connections with other branches of mathematics and computer science: algorithms, computer-drawing, group theory, Riemann surfaces, enumerative combinatorics, block designs, finite geometries, Euclidean and non-Euclidean geometry, knot theory, the absolute Galois group, C^*-algebras, and even string theory.

This volume attempts to survey the principal results within over-arching themes for the myriad aspects of topological graph theory. The authors of the chapters are recognized authorities in their fields. This book is written for the non-specialist and can be used as the basis for a graduate-level course. Nonetheless, the individual chapters cover their fields in great depth and detail, so that even specialists will find the book valuable, both as a reference and as a source of new insights and problems.

JONATHAN L. GROSS
THOMAS W. TUCKER

Preface

The field of graph theory has undergone tremendous growth during the past century. As recently as fifty years ago, the graph theory community had few members and most were in Europe and North America; today there are hundreds of graph theorists and they span the globe. By the mid-1970s, the field had reached the point where we perceived the need for a collection of surveys of the areas of graph theory: the result was our three-volume series *Selected Topics in Graph Theory,* comprising articles written by distinguished experts in a common style. During the past quarter-century, the transformation of the subject has continued, with individual areas (such as topological graph theory) expanding to the point of having important sub-branches themselves. This inspired us to conceive of a new series of books, each a collection of articles within a particular area written by experts within that area. The first of these books was our companion volume on algebraic graph theory, published in 2004. This is the second of these books.

One innovative feature of these volumes is the engagement of academic consultants (here, Jonathan Gross and Thomas Tucker) to advise us on topics to be included and authors to be invited. We believe that this has been successful, the result being chapters covering the full range of areas within topological graph theory written by authors from around the world. Another important feature is that we have imposed uniform terminology and notation throughout, as far as possible, in the belief that this will aid readers in going from one chapter to another. For a similar reason we have not attempted to remove a small amount of overlap between the various chapters.

We hope that these features will make the book easier to use in an advanced course or seminar. We heartily thank the authors for cooperating on this, even though it sometimes required their abandoning some of their favourite conventions – for example, many mathematicians use χ to denote the Euler characteristic, whereas for graph theorists χ usually denotes the chromatic number: the graph theorists won on this one. We also asked our contributors to undergo

the ordeal of having their early versions subjected to detailed critical reading. We believe that the final product is thereby significantly better than it might otherwise have been, simply a collection of individually authored chapters. We want to express our sincere appreciation to all of our contributors for their cooperation.

We extend special thanks to Jonathan Gross and Thomas Tucker for their willingness to share their expertise as academic consultants – their advice has been invaluable. We are also grateful to Cambridge University Press for publishing this work; in particular, we thank Roger Astley and Clare Dennison for their advice, support and cooperation. Finally, we extend our appreciation to several universities for the different ways in which they have assisted with this endeavour: the first editor is grateful to Indiana University–Purdue University in Fort Wayne, while the second editor has had the cooperation of the Open University and Keble College, Oxford.

LOWELL W. BEINEKE
ROBIN J. WILSON