Introduction

LOWELL W. BEINEKE and ROBIN J. WILSON

1. Graph theory

This section presents the basic definitions, terminology and notation of graph theory, along with some fundamental results. Further information can be found in the many standard books on the subject – for example, Chartrand and Lesniak [1], Gross and Yellen [2], West [3] or (for a simpler treatment) Wilson [4].

Graphs

A *graph* G is a pair of sets (V, E), where V is a finite non-empty set of elements called *vertices*, and E is a finite set of elements called *edges*, each of which has two associated vertices (which may be the same). The sets V and E are the *vertex-set* and *edge-set* of G, and are sometimes denoted by $V(G)$ and $E(G)$. The *order* of G is the number of vertices, usually denoted by n, and the number of edges is denoted by m.

An edge whose vertices coincide is called a *loop*, and if two vertices are joined by more than one edge, these are called *multiple edges*. A graph with no loops or multiple edges is a *simple graph*. In many areas of graph theory there is little need for graphs that are not simple, in which case an edge e can be considered as a pair of vertices, $e = \{v, w\}$, or vw for simplicity. However, in topological graph theory, it is often useful, and sometimes necessary, to allow loops and multiple

Fig. 1.

edges. A graph of order 4 and its underlying simple graph are shown in Fig. 1. The *complement* $\overline{G}$ of a simple graph G has the same vertices as G, but two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G.

Adjacency and degrees

The vertices of an edge are *incident* with the edge, and the edge is said to *join* these vertices. Two vertices that are joined by an edge are *neighbours* and are said to be *adjacent*. The set $N(v)$ of neighbours of a vertex v is its *neighbourhood*. Two edges are *adjacent* if they have a vertex in common.

The *degree* deg v of a vertex v is the number of times that it occurs as an endpoint of an edge (with a loop counted twice); in a simple graph, the degree of a vertex is just the number of its neighbours. A vertex of degree 0 is an *isolated vertex* and one of degree 1 is an *end vertex*. A graph is *regular* if all of its vertices have the same degree, and is *k-regular* if that degree is k; a 3-regular graph is sometimes called *cubic*. The maximum degree in a graph G may be denoted by Δ or $\Delta(G)$, and the minimum degree by δ or $\delta(G)$.

Isomorphisms and automorphisms

An *isomorphism* between two graphs G and H consists of a pair of bijections, one between their vertex-sets and the other between their edge-sets, that preserve incidence and non-incidence. (For simple graphs, this amounts to having a bijection between their vertex-sets that preserves adjacency and non-adjacency.) The graphs G and H are *isomorphic*, denoted by $G \approx H$ or $G \cong H$, if there exists an isomorphism between them.

An *automorphism* of a graph G is an isomorphism of G with itself. The set of all automorphisms of a graph G forms a group, called the *automorphism group* of G and denoted by Aut(G). A graph G is *vertex-transitive* if, for any vertices v and w, there is an automorphism taking v to w. It is *edge-transitive* if, for any edges e and f, there is an automorphism taking the vertices of e to those of f. It is *arc-transitive* if, given two ordered pairs of adjacent vertices (v, w) and

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(v', w'), there is an automorphism taking v to v' and w to w'. This is stronger than edge-transitivity, since it implies that for each edge there is an automorphism that interchanges its vertices.

Walks, paths and cycles

A *walk* in a graph is a sequence of vertices and edges $v_0, e_1, v_1, \ldots, e_k, v_k$, in which each edge e_i joins the vertices v_{i-1} and v_i. This walk *goes from* v_0 *to* v_k or *connects* v_0 *and* v_k, and is called a v_0-v_k walk. For simple graphs, it is frequently shortened to $v_0 v_1 \cdots v_k$, since the edges can be inferred from this. Its *length* is k, the number of occurrences of edges, and if $v_0 = v_k$, the walk is *closed*. Some important types of walk are the following:

- a *path* is a walk in which no vertex is repeated;
- a *cycle* is a non-trivial closed walk in which no vertex is repeated, except the first and last;
- a *trail* is a walk in which no edge is repeated;
- a *circuit* is a non-trivial closed trail.

Connectedness and distance

A graph is *connected* if there is a path connecting each pair of vertices, and *disconnected* otherwise. A (*connected*) *component* of a graph is a maximal connected subgraph.

In a connected graph, the *distance* $d(v, w)$ from v to w is the length of a shortest v-w path. It is easy to see that distance satisfies the properties of a metric.

The *diameter* of a connected graph G is the maximum distance between two vertices of G. If G has a cycle, the *girth* of G is the length of a shortest cycle.

A connected graph is *Eulerian* if it has a closed trail containing all the edges of G; such a trail is an *Eulerian trail*. The following are equivalent for a connected graph G:

- G is Eulerian;
- the degree of each vertex of G is even;
- the edge-set of G can be partitioned into cycles.

A graph is *Hamiltonian* if it has a spanning cycle, and is *traceable* if it has a spanning path. No ‘good’ characterizations of these properties are known.

Bipartite graphs and trees

If the set of vertices of a graph G can be partitioned into two non-empty subsets so that no edge joins two vertices in the same subset, then G is *bipartite*. The two subsets are called *partite sets* and, if they have orders r and s, G is sometimes called an $r \times s$ *bipartite graph*. (For convenience, the graph with one vertex and no edges is also called bipartite.) Bipartite graphs are characterized by having no cycles of odd length.

Among the bipartite graphs are *trees*, those connected graphs with no cycles. Trees have been characterized in many ways, some of which we give here. For a graph of order n, the following statements are equivalent:

- G is connected and has no cycles;
- G is connected and has $n - 1$ edges;
- G has no cycles and has $n - 1$ edges;
- G has exactly one path between any two vertices.

A graph without cycles is called a *forest*; thus, each component of a forest is a tree.

The set of trees can also be defined inductively: a single vertex is a tree; and for $n \geq 1$, the trees with $n+1$ vertices are those graphs obtainable from some tree with n vertices by adding a new vertex adjacent to precisely one of its vertices.

This definition has a natural extension to higher dimensions. The *k-dimensional trees*, or *k-trees* for short, are defined as follows. The complete graph on k vertices is a k-tree, and for $n \geq k$, the k-trees with $n+1$ vertices are those graphs obtainable from some k-tree with n vertices by adding a new vertex adjacent to k mutually adjacent vertices in the k-tree. Fig. 2 shows a tree and a 2-tree. An important concept in the study of graph minors (introduced later) is the *tree-width* of a graph G, the minimum dimension of any k-tree that contains G as a subgraph.

Fig. 2.

Special graphs

We now introduce some individual types of simple graph:

- the *complete graph* K_n has n vertices, each adjacent to all the others;
- the *path graph* P_n consists of the vertices and edges of a path of length $n - 1$;
- the *cycle graph* C_n consists of the vertices and edges of a cycle of length n;

- the *complete bipartite graph* $K_{r,s}$ is the simple $r \times s$ bipartite graph in which each vertex is adjacent to every vertex in the other partite set;
- in the *complete k-partite graph* $K_{r_1,r_2,\ldots,r_k}$ the vertices are in k sets with orders $r_1, r_2, \ldots, r_k$, and each vertex is adjacent to every vertex in another set; if the k sets all have order r, the graph is denoted by $K_{k(r)}$.

Examples of these graphs are given in Fig. 3.

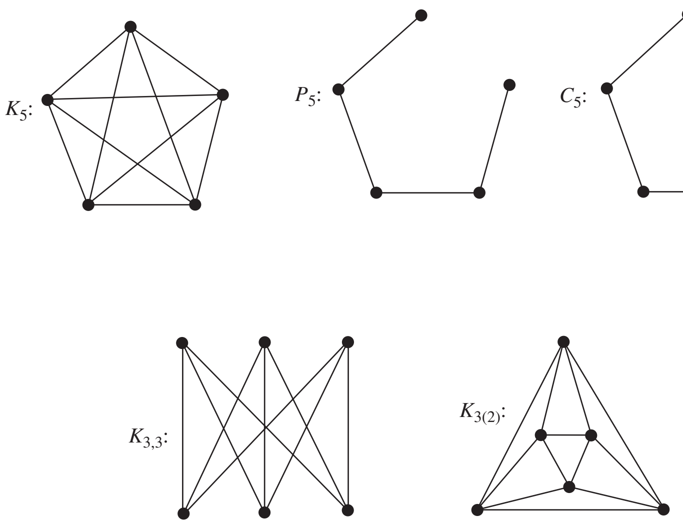

Fig. 3.

We also introduce some special graphs that are not simple:

- the *bouquet* B_m has one vertex and m incident loops;
- the *dipole* D_m consists of two vertices with m edges joining them;
- the *cobblestone path* is the 4-regular graph obtained from the path P_n by doubling each edge and adding a loop at each end.

Fig. 4 gives examples of these graphs.

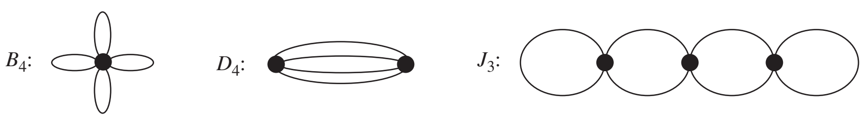

Fig. 4.

A *necklace* is any graph obtained from a cycle by doubling each edge in an independent subset of its edges and adding a loop at each vertex that is not on

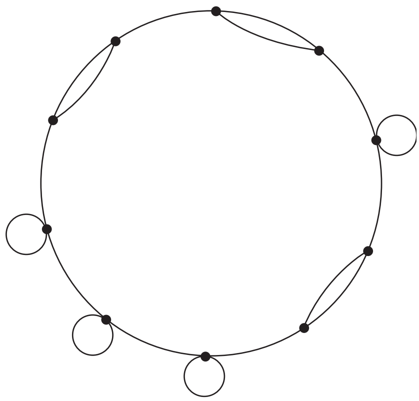

Fig. 5.

one of those edges. It is of *type* (r, s) if it has r doubled edges and s loops (so the original cycle has length $2r + s$). The necklace in Fig. 5 is of type (3, 4).

Operations

Let G and H be graphs with disjoint vertex-sets $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $V(H) = \{w_1, w_2, \ldots, w_r\}$:

- the *union* $G \cup H$ has vertex-set $V(G) \cup V(H)$ and edge-set $E(G) \cup E(H)$. The union of k graphs isomorphic to G is denoted by kG;
- the *join* $G + H$ is obtained from $G \cup H$ by adding an edge from each vertex in G to each vertex in H;
- the *Cartesian product* $G \square H$ (or $G \times H$) has vertex-set $V(G) \times V(H)$, and (v_i, w_j) is adjacent to (v_h, w_k) if either v_i is adjacent to v_h in G and $w_j = w_k$, or $v_i = v_h$ and w_j is adjacent to w_k in H: in less formal terms, $G \square H$ can be obtained by taking n copies of H and joining corresponding vertices in different copies whenever there is an edge in G;
- the *lexicographic product* (or *composition*) $G[H]$ also has vertex-set $V(G) \times V(H)$, but with (v_i, w_j) adjacent to (v_h, w_k) if either v_i is adjacent to v_h in G or $v_i = v_h$ and w_j is adjacent to w_k in H.

Examples of these binary operations are given in Fig. 6.

Subgraphs and minors

If G and H are graphs with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then H is a *subgraph* of G; if, moreover, $V(H) = V(G)$, then H is a *spanning subgraph*. The

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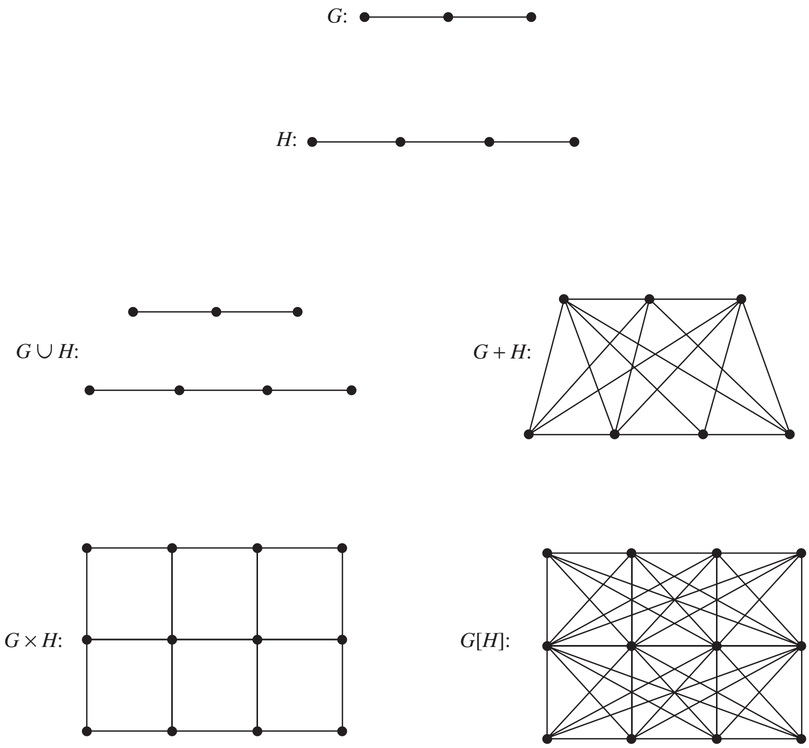

Fig. 6.

subgraph $\langle S \rangle$ *induced* by a non-empty set of S of vertices of G is the subgraph H whose vertex-set is S and whose edge-set consists of those edges of G that join two vertices in S. A subgraph H of G is an *induced subgraph* if $H = \langle V(H) \rangle$. In Fig. 7, H_1 is a spanning subgraph of G, and H_2 is an induced subgraph.

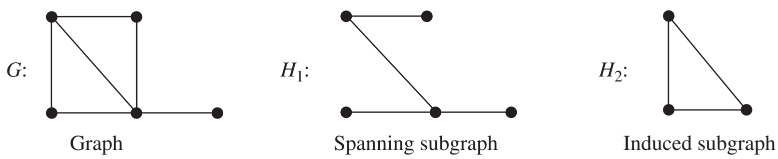

Fig. 7.

Given a graph G, the *deletion of a vertex* v results in the subgraph obtained by removing v and all of its incident edges; it is denoted by $G - v$ and is the subgraph induced by $V - \{v\}$. More generally, if S is any set of vertices in G, then $G - S$ is the graph obtained from G by deleting all the vertices in S and their incident

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edges – that is, $G - S = \langle V - S \rangle$. Similarly, the *deletion of an edge e* results in the subgraph $G - e$ and, for any set X of edges, $G - X$ is the graph obtained from G by deleting all the edges in X.

There are two other operations that are especially important for topological graph theory. If an edge e joins vertices v and w, the *subdivision* of e replaces e by a new vertex u and two new edges vu and uw. Two graphs are *homeomorphic* if there is some graph from which each can be obtained by a sequence of subdivisions. The *contraction* of e replaces the vertices v and w of e by a new vertex u, with an edge ux for each edge vx or wx in G. The operations of subdivision and contraction are illustrated in Fig. 8.

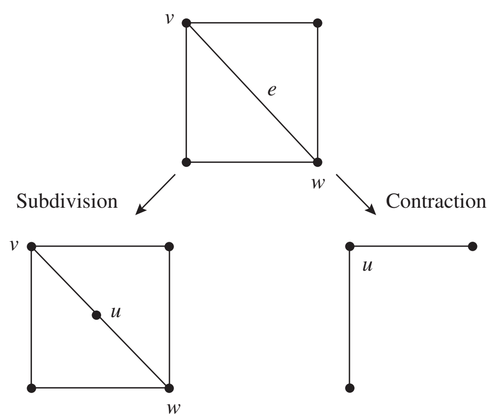

Fig. 8.

If a graph H can be obtained from G by sequence of edge-contractions and the removal of isolated vertices, then G is *contractible* to H. A *minor* of G is any graph that can be obtained from G by a sequence of edge-deletions and edge-contractions, along with deletions of isolated vertices.

Connectedness and connectivity

A vertex v of G is a *cut-vertex* if $G - v$ has more components than G. A non-trivial connected graph with no cut-vertices is 2-*connected* or *non-separable*. The following statements are equivalent for a graph G with at least three vertices:

- G is non-separable;
- every pair of vertices lie on a cycle;
- for any three vertices u, v and w, there is a u-w path containing v;
- for any three vertices u, v and w, there is a u-w path not containing v.

More generally, G is *k-connected* if there is no set S of fewer than k vertices for which $G-S$ is a connected non-trivial graph. Menger gave a useful characterization of such graphs:

Menger's theorem (vertex version) *A graph G is k-connected if and only if, for each pair of vertices v and w, there is a set of k internally disjoint v-w paths.*

The *connectivity* $\kappa(G)$ of a graph G is the maximum value of k for which G is k-connected.

There are similar concepts and results for edges. A *cut-edge* (or *bridge*) is an edge whose deletion produces one more component than before. (Note: for some authors, 'bridge' has a different meaning.) A non-trivial graph is *k-edge-connected* if the result of removing fewer than k edges is always connected, and the *edge-connectivity* $\lambda(G)$ is the maximum value of k for which G is k-edge-connected. We note that Menger's theorem also has an edge version:

Menger's Theorem (edge version) *A graph G is k-edge-connected if and only if, for each pair of vertices v and w, there is a set of k edge-disjoint v-w paths.*

Graph colourings

A graph is *k-colourable* if, from a set of k colours, it is possible to assign a colour to each vertex in such a way that adjacent vertices always have different colours. The *chromatic number* $\chi(G)$ is the least value of k for which G is *k-colourable*, and if $\chi(G) = k$, then G is *k-chromatic*. It is easy to see that a graph is 2-colourable if and only if it is bipartite, but there is no 'good' way to determine which graphs are k-colourable, for any $k \geq 3$. Brooks's theorem provides one of the best-known bounds on the chromatic number of a graph.

Brooks's theorem *If G is a simple graph with maximum degree Δ and is neither an odd cycle nor a complete graph, then $\chi(G) \leq \Delta$.*

There are similar concepts for colouring edges. A graph is *k-edge-colourable* if, from a set of k colours, it is possible to assign a colour to each edge in such a way that adjacent edges always have different colours. The *chromatic index* $\chi'(G)$ is the least value of k for which G is k-edge-colourable. Vizing [11] proved that the range of values of $\chi'(G)$ is quite limited:

Vizing's theorem *If G is a simple graph with maximum degree Δ, then $\chi'(G) = \Delta$ or $\Delta + 1$.*

Directed graphs

Digraphs are directed analogues of graphs, and thus have many similarities, as well as some important differences.

A *digraph* (or *directed graph*) D is a pair of sets (V, E), where V is a finite non-empty set of elements called *vertices*, and E is a set of ordered pairs of distinct elements of V called *arcs* or *directed edges*. Note that the elements of E are ordered, which gives each of them a direction. An example of a digraph is shown in Fig. 9.

D:

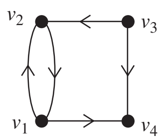

Fig. 9.

Because of the similarities between graphs and digraphs, we mention only the main differences here and do not redefine those concepts that carry over easily. An arc (v, w) in a simple digraph may be written as vw, and is said to *go from* v *to* w, or to *go out of* v *and into* w. In the context of digraphs, walks, paths, cycles, trails and circuits are understood to be directed, unless otherwise indicated.

A digraph D is *strongly connected* if there is a path from each vertex to each of the others. A *strong component* is a maximal strongly connected subgraph. Connectivity and edge-connectivity are defined in terms of strong connectedness.

2. Graphs in the plane

In this section we briefly survey properties of graphs that can be drawn in the plane without any edges crossing. To make this more precise, we define an *embedding* of a graph G *in the plane* to be a one-to-one mapping of the vertices of G into the plane and a mapping of the edges of G to disjoint simple open arcs, so that the image of each edge joins the images of its two vertices and none of the images of the edges contains the image of a vertex.

Here there is little to be gained by allowing loops or multiple edges, so in this section we assume that all graphs are simple. A graph that can be embedded in the plane is called a *planar graph*, and its image is called a *plane graph*. An example is given in Fig. 10.

A *region* of an embedded graph G is a maximal connected set of points in the relative complement of G in the plane; note that one region is unbounded. The topological closure of a region (that is, the region together with the vertices and