Topics in Algebraic Graph Theory

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CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK
40 West 20th Street, New York, NY 10011-4211, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa

http://www.cambridge.org

First published 2005

Printed in the United States of America

Typeface Times Roman 10/13 pt. *System* LaTeX 2_ε [TB]

A catalog record for this book is available from the British Library.

Library of Congress Cataloging in Publication Data

Topics in algebraic graph theory / edited by Lowell W. Beineke and Robin J. Wilson,
academic consultant, Peter J. Cameron.

p. cm. – (Encyclopedia of mathematics and its applications)

Includes bibliographical references and index.

ISBN 0-521-80197-4

1. Graph theory. I. Beineke, Lowell W. II. Wilson, Robin J. III. Series.

QA166.T64 2004
511′.5 – dc22 2004045915

ISBN 0 521 80197 4 hardback

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Introduction

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This introductory chapter is divided into three parts. The first presents the basic ideas of graph theory. The second concerns linear algebra (for Chapters 1–4), while the third concerns group theory (for Chapters 5–10).

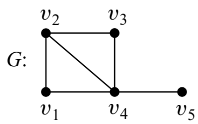
1. Graph theory

This section presents the basic definitions, terminology and notations of graph theory, along with some fundamental results. Further information can be found in the many standard books on the subject – for example, West [4] or (for a simpler treatment) Wilson [5].

Graphs

A *graph* G is a pair of sets (V, E), where V is a finite non-empty set of elements called *vertices*, and E is a set of unordered pairs of distinct vertices called *edges*. The sets V and E are the *vertex-set* and the *edge-set* of G, and are often denoted by $V(G)$ and $E(G)$, respectively. An example of a graph is shown in Fig. 1.

The number of vertices in a graph is the *order* of the graph; usually it is denoted by n and the number of edges by m. Standard notation for the vertex-set is $V = \{v_1, v_2, \ldots, v_n\}$ and for the edge-set is $E = \{e_1, e_2, \ldots, e_m\}$. Arbitrary vertices are frequently represented by $u, v, w, \ldots$ and edges by $e, f, \ldots$.



$V = \{v_1, v_2, v_3, v_4, v_5\}$

$E = \{v_1v_2, v_1v_4, v_2v_3, v_2v_4, v_3v_4, v_4v_5\}$

Fig. 1.

Variations of graphs

By definition, our graphs are *simple*, meaning that two vertices are connected by at most one edge. If several edges, called *multiple edges*, are allowed between two vertices, we have a *multigraph*. Sometimes, *loops* – edges joining vertices to themselves – are also permitted. In a *weighted graph*, the edges are assigned numerical values called *weights*. Finally, if the vertex-set is allowed to be infinite, then G is an *infinite graph*.

Perhaps the most important variation is that of *directed graphs*; these are discussed at the end of this section.

Adjacency and degrees

For convenience, the edge $\{v, w\}$ is commonly written as vw. We say that this edge *joins* v and w and that it is *incident with* v and w. In this case, v and w are *adjacent vertices*, or *neighbours*. The set of neighbours of a vertex v is its *neighbourhood* $N(v)$. Two edges are *adjacent edges* if they have a vertex in common.

The number of neighbours of a vertex v is called its *degree*, denoted by deg v. Observe that the sum of the degrees in a graph is twice the number of edges. If all the degrees of G are equal, then G is *regular*, or is *k-regular* if that common degree is k. The maximum degree in a graph is often denoted by Δ.

Walks

A *walk* in a graph is a sequence of vertices and edges $v_0, e_1, v_1, \ldots, e_k, v_k$, in which each edge $e_i = v_{i-1}v_i$. This walk goes *from* v_0 *to* v_k or *connects* v_0 and v_k, and is called a v_0-v_k walk. It is frequently shortened to $v_0v_1 \ldots v_k$, since the edges may be inferred from this. Its *length* is k, the number of occurrences of edges. If $v_k = v_0$, the walk is *closed*.

Some important types of walk are the following:

- a *path* is a walk in which no vertex is repeated;
- a *trail* is a walk in which no edge is repeated;
- a *cycle* is a non-trivial closed trail in which no vertex is repeated.

Distance

In a connected graph, the *distance* between two vertices v and w is the minimum length of a path from v to w, and is denoted by $d(v, w)$. It is easy to see that distance satisfies the properties of a metric: for all vertices u, v and w,

- $d(v, w) \geq 0$, with equality if and only if $v = w$;
- $d(v, w) = d(w, v)$;
- $d(u, w) \leq d(u, v) + d(v, w)$

The *diameter* of a graph G is the maximum distance between two vertices of G. If G has cycles, the *girth* of G is the length of a shortest cycle, and the *circumference* is the length of a longest cycle.

Subgraphs

If G and H are graphs with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then H is a *subgraph* of G. If, moreover, $V(H) = V(G)$, then H is a *spanning* subgraph. The subgraph *induced* by a non-empty set S of vertices in G is that subgraph H with vertex-set S whose edge-set consists of those edges of G that join two vertices in S; it is denoted by $\langle S \rangle$ or $G[S]$. A subgraph H of G is *induced* if $H = \langle V(H) \rangle$. In Fig. 2, H_1 is a spanning subgraph of G, and H_2 is an induced subgraph.

Given a graph G, the *deletion of a vertex* v results in the subgraph obtained by excluding v and all edges incident with it. It is denoted by $G - v$ and is the subgraph induced by $V - \{v\}$. More generally, if $S \subset V$, we write $G - S$ for the graph obtained from G by deleting all of the vertices of S; that is, $G - S = \langle V - S \rangle$.

The *deletion of an edge* e results in the subgraph $G - e$ obtained by excluding e from E; for $F \subseteq E$, $G - F$ denotes the spanning subgraph with edge-set $E - F$.

Connectedness and connectivity

A graph G is *connected* if there is a path connecting each pair of vertices. A *(connected) component* of G is a maximal connected subgraph of G.

A vertex v of a graph G is a *cut-vertex* if $G - v$ has more components than G. A connected graph with no cut-vertices is 2-*connected* or *non-separable*. The following statements are equivalent for a graph G with at least three vertices:

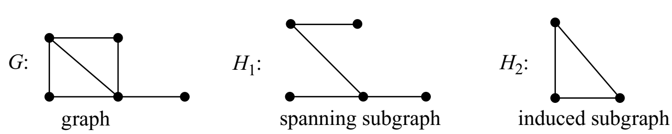


Fig. 2.

- G is non-separable;
- every pair of vertices lie on a cycle;
- every vertex and edge lie on a cycle;
- every pair of edges lie on a cycle;
- for any three vertices u, v, and w, there is a v-w path containing u;
- for any three vertices u, v, and w, there is a v-w path not containing u;
- for any two vertices v and w and any edge e, there is a v-w path containing e.

More generally, a graph G is *k-connected* if there is no set S with fewer than k vertices for which $G - S$ is a connected non-trivial graph. Menger characterized such graphs.

Menger's theorem *A graph G is k-connected if and only if, for each pair of vertices v and w, there is a set of k v-w paths that pairwise have only v and w in common.*

The *connectivity* $\kappa(G)$ of a graph G is the maximum value of k for which G is k-connected.

There are similar concepts and results for edges. A *cut-edge* (or *bridge*) is any edge whose deletion produces one more component than before. A non-trivial graph G is *k-edge-connected* if the result of removing fewer than k edges is always connected, and the *edge-connectivity* $\lambda(G)$ is the maximum value of k for which G is k-edge-connected. We note that Menger's theorem also has an edge version.

Bipartite graphs

If the vertices of a graph G can be partitioned into two non-empty sets so that no edge joins two vertices in the same set, then G is *bipartite*. The two sets are called *partite sets*, and if they have orders r and s, G may be called an $r \times s$ bipartite graph. The most important property of bipartite graphs is that they are the graphs that contain no cycles of odd length.

Trees

A *tree* is a connected graph that has no cycles. They have been characterized in many ways, a few of which we give here. For a graph G of order n:

- G is connected and has no cycles;
- G is connected and has $n - 1$ edges;
- G has no cycles and has $n - 1$ edges.

Any graph without cycles is a *forest*; note that each component of a forest is a tree.

Special graphs

We now introduce some individual types of graphs:

- the *complete graph* K_n has n vertices, each of which is adjacent to all of the others;
- the *null graph* N_n has n vertices and no edges;
- the *path graph* P_n consists of the vertices and edges of a path of length $n - 1$;
- the *cycle graph* C_n consists of the vertices and edges of a cycle of length n;
- the *complete bipartite graph* $K_{r,s}$ is the $r \times s$ bipartite graph in which each vertex is adjacent to all those in the other partite set;
- in the *complete k-partite graph*, $K_{r_1,r_2,\ldots,r_n}$ the vertices are in k sets (having orders $r_1, r_2, \ldots, r_k$) and each vertex is adjacent to all the others, except those in the same set. If the k sets all have order r, the graph is denoted by $K_{k(r)}$. The graph $K_{k(2)}$ is sometimes called the *k-dimensional octahedral graph* or *cocktail party graph*, also denoted by CP(k); $K_{3(2)}$ is the graph of an octahedron.
- the *d-dimensional cube* (or *d-cube*) Q_d is the graph whose vertices can be labelled with the 2^d binary d-tuples, in such a way that two vertices are adjacent when their labels differ in exactly one position. It is regular of degree d, and is isomorphic to the lattice of subgraphs of a set of d elements.

Examples of these graphs are given in Fig. 3.

Operations on graphs

There are several ways to get new graphs from old. We list some of the most important here.

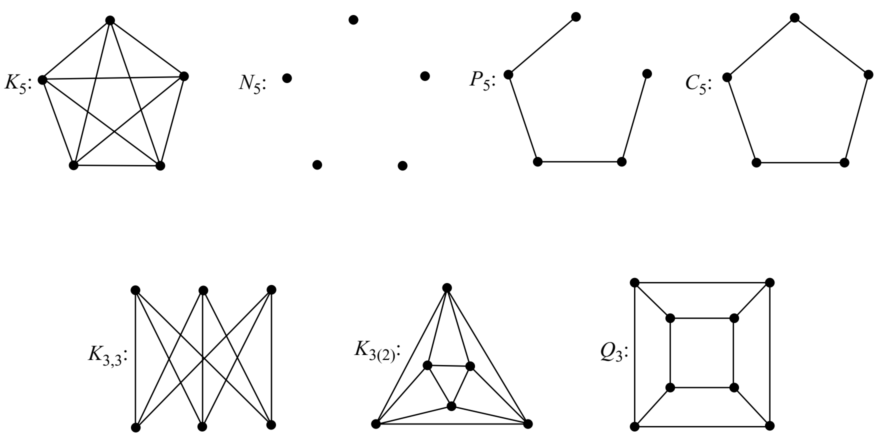


Fig. 3.

- The *complement* $\overline{G}$ of a graph G has the same vertices as G, but two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G.

For the other operations, we assume that G and H are graphs with disjoint vertex-sets, $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $V(H) = \{w_1, w_2, \ldots, w_t\}$:

- the *union* $G \cup H$ has vertex-set $V(G) \cup V(H)$ and edge-set $E(G) \cup E(H)$. The union of k graphs isomorphic to G is denoted by kG.
- the *join* $G + H$ is obtained from $G \cup H$ by adding all of the edges from vertices in G to those in H.
- the *(Cartesian) product* $G \square H$ or $G \times H$ has vertex-set $V(G) \times V(H)$, and (v_i, w_j) is adjacent to (v_h, w_k) if either (a) v_i is adjacent to v_h in G and $w_j = w_k$, or (b) $v_i = v_h$ and w_j is adjacent to w_k in H. In less formal terms, $G \square H$ can be obtained by taking n copies of H and joining corresponding vertices in different copies whenever there is an edge in G. Note that, for d-cubes, $Q_{d+1} = K_2 \square Q_d$ (with $Q_1 = K_2$).

Examples of these binary operations are given in Fig. 4.

There are two basic operations involving an edge of a graph. The *insertion of a vertex into an edge e* means that the edge $e = vw$ is replaced by a new vertex u and the two edges vu and uw. Two graphs are *homeomorphic* if each can be obtained from a third graph by a sequence of vertex insertions. The *contraction of the edge vw* means that v and w are replaced by a new vertex u that is adjacent to the other neighbours of v and w. If a graph H can be obtained from G by a sequence of edge contractions and the deletion of isolated vertices, then G is said to be *contractible* to H. Finally, H is a *minor* of G if it can be obtained from G by a sequence of edge-deletions and edge-contractions and the removal

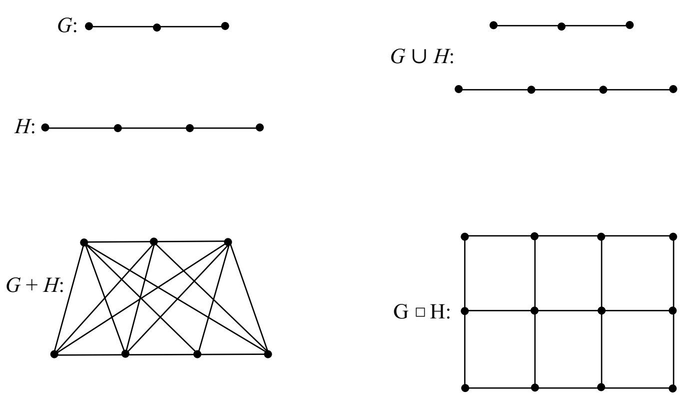


Fig. 4.

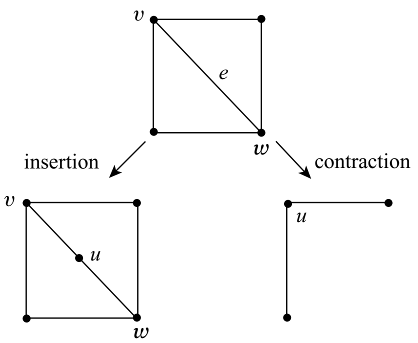


Fig. 5.

of isolated vertices. The operations of insertion and contraction are illustrated in Fig. 5.

Traversability

A connected graph G is *Eulerian* if it has a closed trail containing all of the edges of G; such a trail is called an *Eulerian trail*. The following are equivalent for a connected graph G:

- G is Eulerian;
- the degree of each vertex of G is even;
- the edge-set of G can be partitioned into cycles.

A graph G is *Hamiltonian* if it has a spanning cycle, and *traceable* if it has a spanning path. No 'good' characterizations of these graphs are known.

Planarity

A *planar graph* is one that can be embedded in the plane in such a way that no two edges meet except at a vertex incident with both. If a graph G is embedded in this way, then the points of the plane not on G are partitioned into open sets called *faces* or *regions*. Euler discovered the basic relationship between the numbers of vertices, edges and faces.

Euler's polyhedron formula *Let G be a connected graph embedded in the plane with n vertices, m edges and f faces. Then* $n - m + f = 2$.

It follows from this result that a planar graph with n vertices ($n \geq 3$) has at most $3(n-2)$ edges, and at most $2(n-2)$ edges if it is bipartite. From this it follows that the two graphs K_5 and $K_{3,3}$ are non-planar. Kuratowski proved that these two graphs are the only barriers to planarity.

Kuratowski's theorem *The following statements are equivalent for a graph G:*
- *G is planar;*
- *G has no subgraph that is homeomorphic to K_5 or $K_{3,3}$;*
- *G has no subgraph that is contractible to K_5 or $K_{3,3}$.*

Graph colourings

A graph G is *k-colourable* if, from a set of k colours, it is possible to assign a colour to each vertex in such a way that adjacent vertices always have different colours. The *chromatic number* $\chi(G)$ is the least value of k for which G is *k-chromatic*. It is easy to see that a graph is 2-colourable if and only if it is bipartite, but there is no 'good' way to determine which graphs are k-colourable for $k \geq 3$. Brooks's theorem provides one of the best-known bounds on the chromatic number of a graph.

Brooks's theorem *If G is a graph with maximum degree Δ that is neither an odd cycle nor a complete graph, then $\chi(G) \leq \Delta$.*

There are similar concepts for colouring edges. A graph G is *k-edge-colourable* if, from a set of k colours, it is possible to assign a colour to each edge in such a way that adjacent edges always have different colours. The *edge-chromatic number* $\chi'(G)$ is the least k for which G is k-edge-colourable. Vizing proved that the range of values of $\chi'(G)$ is very limited.

Vizing's theorem *If G is a graph with maximum degree Δ, then*

$$\Delta \leq \chi'(G) \leq \Delta + 1.$$

Line graphs

The *line graph* $L(G)$ of a graph G has the edges of G as its vertices, with two of these vertices adjacent if and only if the corresponding edges are adjacent in G. An example is given in Fig. 6.

A graph is a line graph if and only if its edges can be partitioned into complete subgraphs in such a way that no vertex is in more than two of these subgraphs.

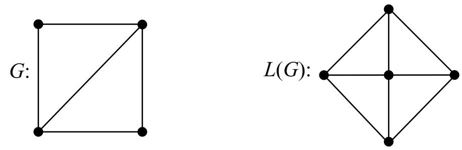
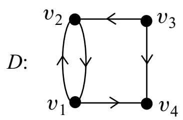


Fig. 6.



$V = \{v_1, v_2, v_3, v_4\}$
$E = \{\overrightarrow{v_1v_2}, \overrightarrow{v_1v_4}, \overrightarrow{v_2v_1}, \overrightarrow{v_3v_2}, \overrightarrow{v_3v_4}\}.$

Fig. 7.

Line graphs are also characterized by the property of having none of nine particular graphs as a forbidden subgraph.

Directed graphs

Digraphs are directed analogues of graphs, and thus have many similarities, as well as some important differences.

A *digraph* (or *directed graph*) D is a pair of sets (V, E) where V is a finite non-empty set of elements called *vertices*, and E is a set of ordered pairs of distinct elements of V called *arcs* or *directed edges*. Note that the elements of E are now ordered, which gives each of them a direction. An example of a digraph is given in Fig. 7.

Because of the similarities between graphs and digraphs, we mention only the main differences here and do not redefine those concepts that carry over easily.

An arc (v, w) of a digraph may be written as $\overrightarrow{vw}$, and is said to go *from* v *to* w, or to *go out of* v and *go into* w.

Walks, paths, trails and cycles are understood to be directed, unless otherwise indicated.

The *out-degree* $d^+(v)$ of a vertex v in a digraph is the number of arcs that go out of it, and the *in-degree* $d^-(v)$ is the number of arcs that go into it.

A digraph D is *strongly connected*, or *strong*, if there is a path from each vertex to each of the others. A *strong component* is a maximal strongly connected subgraph. Connectivity and edge-connectivity are defined in terms of strong connectedness.

A *tournament* is a digraph in which every pair of vertices are joined by exactly one arc. One interesting aspect of tournaments is their Hamiltonian properties:

- every tournament has a spanning path;
- a tournament has a Hamiltonian cycle if and only if it is strong.

2. Linear algebra

In this section we present the main results on vector spaces and matrices that are used in Chapters 1–4. For further details, see [3].

The space $\mathbf{R}^n$

The *real n-dimensional space* $\mathbf{R}^n$ consists of all n-tuples of real numbers $\mathbf{x} = (x_1, x_2, \ldots, x_n)$; in particular, the plane $\mathbf{R}^2$ consists of all pairs (x_1, x_2), and three-dimensional space $\mathbf{R}^3$ consists of all triples (x_1, x_2, x_3). The elements $\mathbf{x}$ are *vectors*, and the numbers x_i are the *coordinates* or *components* of $\mathbf{x}$.

When $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ are vectors in $\mathbf{R}^n$, we can form their *sum* $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$, and if α is a scalar (real number), we can form the *scalar multiple* $\alpha\mathbf{x} = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n)$.

The *zero vector* is the vector $\mathbf{0} = (0, 0, \ldots, 0)$, and the *additive inverse* of $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is the vector $-\mathbf{x} = (-x_1, -x_2, \ldots, -x_n)$.

We can similarly define the *complex n-dimensional space* $\mathbf{C}^n$, in which the vectors are all n-tuples of *complex* numbers $\mathbf{z} = (z_1, z_2, \ldots, z_n)$; in this case, we take the multiplying scalars α to be complex numbers.

Metric properties

When $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ are vectors in $\mathbf{R}^n$, their *dot product* is the scalar $\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n$. The dot product is sometimes called the *inner product* and denoted by $\langle \mathbf{x}, \mathbf{y} \rangle$.

The *length* or *norm* $\|\mathbf{x}\|$ of a vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is

$$(\mathbf{x} \cdot \mathbf{x})^{1/2} = \left(x_1^2 + x_2^2 + \cdots + x_n^n\right)^{1/2}.$$

A *unit vector* is a vector $\mathbf{u}$ for which $\|\mathbf{u}\| = 1$, and for any non-zero vector $\mathbf{x}$, the vector $\mathbf{x}/\|\mathbf{x}\|$ is a unit vector.

When $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$, the *distance* between $\mathbf{x}$ and $\mathbf{y}$ is $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$. The distance function d satisfies the usual properties of a metric: *for any* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{R}^n$,

- $d(\mathbf{x}, \mathbf{y}) \geq 0$, *and* $d(\mathbf{x}, \mathbf{y}) = 0$ *if and only if* $\mathbf{x} = \mathbf{y}$;
- $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$;
- $d(\mathbf{x}, \mathbf{z}) \leq d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})$ (*triangle inequality*).

The following result is usually called the *Cauchy-Schwarz inequality*:

Cauchy-Schwarz inequality *For any* $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$, $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \cdot \|\mathbf{y}\|$.

We define the angle θ between the non-zero vectors $\mathbf{x}$ and $\mathbf{y}$ by

$$\cos\theta = \mathbf{x}\cdot\mathbf{y}/\|\mathbf{x}\|\cdot\|\mathbf{y}\|.$$

Two vectors $\mathbf{x}$ and $\mathbf{y}$ are *orthogonal* if the angle between them is 0 – that is, if $\mathbf{x}\cdot\mathbf{y} = 0$. In this case, we have the following celebrated result.

Pythagoras's theorem *If* $\mathbf{x}$ *and* $\mathbf{y}$ *are orthogonal, then* $\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$.

An *orthogonal set of vectors* is a set of vectors each pair of which is orthogonal. An *orthonormal set* is an orthogonal set in which each vector has length 1.

In a complex space $\mathbf{C}^n$ most of the above concepts are defined as above. One exception is that the dot product of two complex vectors $\mathbf{z} = (z_1, z_2, \ldots, z_n)$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ is now defined by $\mathbf{z}\cdot\mathbf{w} = z_1\overline{w}_1 + z_2\overline{w}_2 + \cdots + z_n\overline{w}_n$, where $\overline{w}$ is the complex conjugate of w.

Vector spaces

A *real vector space* V is a set of elements, called *vectors*, with rules of addition and scalar multiplication that satisfy the following conditions:

Addition
A1: For all $\mathbf{x}, \mathbf{y} \in V$, $\mathbf{x}+\mathbf{y} \in V$;
A2: For all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, $(\mathbf{x}+\mathbf{y})+\mathbf{z} = \mathbf{x}+(\mathbf{y}+\mathbf{z})$;
A3: There is an element $\mathbf{0} \in V$ *satisfying* $\mathbf{x}+\mathbf{0} = \mathbf{x}$, *for all* $\mathbf{x} \in V$;
A4: For each $\mathbf{x} \in V$, *there is an element* $-\mathbf{x}$ *satisfying* $\mathbf{x}+(-\mathbf{x}) = \mathbf{0}$;
A5: For all $\mathbf{x}, \mathbf{y} \in \mathbf{V}$, $\mathbf{x}+\mathbf{y} = \mathbf{y}+\mathbf{x}$.

Scalar multiplication
M1: For all $\mathbf{x} \in V$ *and* $\alpha \in \mathbf{R}$, $\alpha\mathbf{x} \in V$;
M2: For all $\mathbf{x} \in V$, $1\mathbf{x} = \mathbf{x}$;
M3: For all $\alpha, \beta \in \mathbf{R}$, $\alpha(\beta\mathbf{x}) = (\alpha\beta)\mathbf{x}$;

Distributive laws
D1: For all $\alpha, \beta \in \mathbf{R}$ *and* $\mathbf{x} \in V$, $(\alpha+\beta)\mathbf{x} = \alpha\mathbf{x}+\beta\mathbf{x}$;
D2: For all $\alpha \in \mathbf{R}$ *and* $\mathbf{x}, \mathbf{y} \in V$, $\alpha(\mathbf{x}+\mathbf{y}) = \alpha\mathbf{x}+\alpha\mathbf{y}$.

Examples of real vector spaces are $\mathbf{R}^n$, $\mathbf{C}^n$, the set of all real polynomials, the set of all real infinite sequences, and the set of all functions $f: \mathbf{R} \to \mathbf{R}$, each with the appropriate definitions of addition and scalar multiplication.

Complex vector spaces are defined similarly, except that the scalars are elements of $\mathbf{C}$, rather than $\mathbf{R}$. More generally, the scalars can come from any field, such as the set $\mathbf{Q}$ of rational numbers, the integers $\mathbf{Z}_p$ modulo p, where p is a prime number, or the finite field $\mathbf{F}_q$, where q is a power of a prime.

Subspaces

A non-empty subset W of a vector space V is a *subspace* of V if W is itself a vector space with respect to the operations of addition and scalar multiplication in V. For example, the subspaces of $\mathbf{R}^3$ are $\{\mathbf{0}\}$, the lines and planes through $\mathbf{0}$, and $\mathbf{R}^3$ itself.

When X and Y are subspaces of a vector space V, their *intersection* $X \cap Y$ is also a subspace of V, as is their *sum* $X + Y = \{x + y : x \in X, y \in Y\}$.

When $V = X + Y$ and $X \cap Y = \{\mathbf{0}\}$, we call V the *direct sum* of X and Y, and write $V = X \oplus Y$.

Bases

Let $S = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ be a set of vectors in a vector space V. Then any vector of the form

$$\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \cdots + \alpha_r\mathbf{x}_r,$$

where $\alpha_1, \alpha_2, \ldots, \alpha_r$ are scalars, is a *linear combination* of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r$. The set of all linear combinations of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r$ is a subspace of V called the *span* of S, denoted by $\langle S\rangle$ or $\langle \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\rangle$. When $\langle S\rangle = V$, the set S *spans* V, or is a *spanning set* for V.

The set $S = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ is *linearly dependent* if one of the vectors $\mathbf{x}_i$ is a linear combination of the others – in this case, there are scalars $\alpha_1, \alpha_2, \ldots, \alpha_r$, *not all zero*, for which

$$\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \cdots + \alpha_r\mathbf{x}_r = \mathbf{0}.$$

The set S is *linearly independent* if it is not linearly dependent – that is,

$$\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \cdots + \alpha_r\mathbf{x}_r = \mathbf{0}$$

holds only when $\alpha_1 = \alpha_2 = \cdots = \alpha_r = \mathbf{0}$.

A *basis* B is a linearly independent spanning set for V. In this case, each vector $\mathbf{x}$ of V can be written as a linear combination of the vectors in B *in exactly one way*; for example, the *standard basis* for $\mathbf{R}^3$ is $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ and a basis for the set of all real polynomials is $\{1, x, x^2, \ldots\}$.

Dimension

A vector space V with a finite basis is *finite-dimensional*. In this situation, any two bases for V have the same number of elements. This number is the *dimension* of V, denoted by dim V; for example, $\mathbf{R}^3$ has dimension 3. The dimension of a subspace of V is defined similarly.

When X and Y are subspaces of V, we have the *dimension theorem*:

$$\dim(X + Y) = \dim X + \dim Y - \dim(X \cap Y).$$

When $X \cap Y = \{\mathbf{0}\}$, this becomes

$$\dim(X \oplus Y) = \dim X + \dim Y.$$

Euclidean spaces

Let V be a real vector space, and suppose that with each pair of vectors $\mathbf{x}$ and $\mathbf{y}$ in V is associated a scalar $\langle \mathbf{x}, \mathbf{y} \rangle$. This is an *inner product* on V if it satisfies the following properties: *for any* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$,

- $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$, *and* $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ *if and only if* $\mathbf{x} = 0$;
- $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$;
- $\langle \alpha\mathbf{x} + \beta\mathbf{y}, \mathbf{z} \rangle = \alpha \langle \mathbf{x}, \mathbf{z} \rangle + \beta \langle \mathbf{y}, \mathbf{z} \rangle$.

The vector space V, together with this inner product, is called a *real inner product space*, or *Euclidean space*. Examples of Euclidean spaces are $\mathbf{R}^3$ with the dot product as inner product, and the space V of real-valued continuous functions on the interval $[-1, 1]$ with the inner product defined for $\mathbf{f}, \mathbf{g}$ in V by $\langle \mathbf{f}, \mathbf{g} \rangle = \int_{-1}^{1} \mathbf{f}(t)\mathbf{g}(t)\, dt$. Analogously to the dot product, we can define the metrical notions of length, distance and angle in any Euclidean space, and we can derive analogues of the Cauchy-Schwarz inequality and Pythagoras's theorem.

An *orthogonal basis* for a Euclidean space is a basis in which any two distinct basis vectors are orthogonal. If, further, each basis vector has length 1, then the basis is an *orthonormal basis*. If V is a Euclidean space, the *orthogonal complement* $W^{\perp}$ of a subspace W is the set of all vectors in V that are orthogonal to all vectors in W – that is,

$$W^{\perp} = \{\mathbf{v} \in V : \langle \mathbf{v}, \mathbf{w} \rangle = 0 \text{ for all } \mathbf{w} \in W\}.$$

Linear transformations

When V and W are real vector spaces, a function $T : V \to W$ is a *linear transformation* if, for all $\mathbf{v}_1, \mathbf{v}_2 \in V$ and $\alpha, \beta \in \mathbf{R}$,

$$T(\alpha\mathbf{v}_1 + \beta\mathbf{v}_2) = \alpha T(\mathbf{v}_1) + \beta T(\mathbf{v}_2).$$

If $V = W$, then T is sometimes called a *linear operator* on V.

The linear transformation T is *onto*, or *surjective*, when $T(V) = W$, and is *one-one*, or *injective*, if $T(\mathbf{v}_1) = T(\mathbf{v}_2)$ only when $\mathbf{v}_1 = \mathbf{v}_2$.

The *image* of T is the subspace of W defined by

$$\mathrm{im}(T) = \{\mathbf{w} \in W : \mathbf{w} = T(\mathbf{v}),\ \text{for some } \mathbf{v} \in V\};$$

note that T is onto if and only if $\mathrm{im}(T) = W$.

The *kernel*, or *null space*, of T is the subspace of V defined by

$$\ker(T) = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}_W\};$$

note that T is one-one if and only if $\ker(T) = \{\mathbf{0}_V\}$.

Defining the *rank* and *nullity* of T by

$$\mathrm{rank}(T) = \dim \mathrm{im}(T) \quad \text{and} \quad \mathrm{nullity}(T) = \dim \ker(T),$$

we obtain the *rank-nullity formula*:

$$\mathrm{rank}(T) + \mathrm{nullity}(T) = \dim V.$$

Algebra of linear transformations

When $S\colon U \to V$ and $T : V \to W$ are linear transformations, we can form their *composition* $T \circ S\colon U \to W$, defined by

$$(T \circ S)(\mathbf{u}) = T(S(\mathbf{u})), \qquad \text{for all } \mathbf{u} \in U.$$

The composition of linear transformations is associative.

The linear transformation $T : V \to W$ is *invertible*, or *non-singular*, if there is a linear transformation T^{-1}, called the *inverse* of T, for which $T^{-1} \circ T$ is the identity transformation on V and $T \circ T^{-1}$ is the identity transformation on W. Note that a linear transformation is invertible if and only if it is one-one and onto.

The matrix of a linear transformation

Let $T : V \to W$ be a linear transformation, let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be a basis for V and let $\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_m\}$ be a basis for W. For each $i = 1, 2, \ldots, n$, we can write

$$T(\mathbf{e}_i) = a_{1i}\mathbf{f}_1 + a_{2i}\mathbf{f}_2 + \cdots + a_{mi}\mathbf{f}_m,$$

for some scalars $a_{1i}, a_{2i}, \ldots, a_{mi}$. The rectangular array of scalars

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$