Harmonic Superspace

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Introductory overview

We start by overviewing the origins, motivations, basic ideas and results of the harmonic superspace (and space) approach. Our major aim here is to give the reader a preliminary impression of the subject before immersion into the main body of the book.

1.1 Brief motivations

It is hardly possible to overestimate the rôle of symmetries in the development of physics. The place they occupy is becoming more and more important every year. The very family of symmetries is getting richer all the time: Besides the old symmetries based on Lie algebras we are now exploiting new kinds of symmetries. These include supersymmetries which mix bosons with fermions and are based on superalgebras, symmetries associated with non-linear algebras of Zamolodchikov's type, symmetries connected to quantum groups, etc. To date, the supersymmetric models have been studied in most detail. They turn out to have quite remarkable features. They open a new era in the search for a unified theory of all interactions including gravity. They help to solve the hierarchy problem in the grand unification theories. For the first time in the history of quantum field theory, supersymmetry has led to the discovery of a class of ultraviolet-finite local four-dimensional field theories. In these finite theories the ultraviolet divergences in the boson and fermion loops 'miraculously' cancel against each other. Supersymmetries underlie the superstring theories, which provide the first consistent scheme for quantization of gravity. The research programs of the leading accelerator laboratories include searches for supersymmetric partners of the known particles (predicted by supersymmetry but not yet discovered).

In view of this impressive development, it is imperative to be able to formulate the supersymmetric theories in a systematic, consistent and clear way. There already exist several reviews [F2, F3, F4, F7, N2, O4, S13, V1] and textbooks [B17, G26, W7, W12] devoted to the simplest kind of supersymmetry, $N = 1$ (i.e., containing one spinor generator in its superalgebra). It was this supersymmetry that was first discovered in the pioneering articles [G38, G39, V4, V5, V6, W8]. The superfield approach appropriate to this case was developed in the 1970s. However, extended supersymmetries (i.e., those containing more than one spinor generator) turned out much more difficult. Each new step in understanding them requires new notions and approaches. Even in the simplest extended $N = 2$ supersymmetry, until 1984 no way to formulate all such theories off shell, in a manifestly supersymmetric form and in terms of unconstrained superfields, was known. Such formulations are preferable not only because of their intrinsic beauty, but also since they provide an efficient technique, in particular, in quantum calculations or in the proof of finiteness. The invention of a new, harmonic superspace [G4, G13] made it possible to develop off-shell unconstrained formulations of all the $N = 2$ supersymmetric theories (matter, Yang–Mills and supergravity) and of $N = 3$ Yang–Mills theory.

$N = 2$ harmonic superspace is standard superspace augmented by the two-dimensional sphere $S^2 \sim SU(2)/U(1)$. In such an enlarged superspace it is possible to introduce a new kind of analyticity, Grassmann harmonic [G4, G13]. This proved to be the key to the adequate off-shell unconstrained formulations, just like chirality [F13], the simplest kind of Grassmann analyticity [G8], is a keystone in $N = 1$ supersymmetry. This new analyticity amounts to the existence of an analytic subspace of harmonic superspace whose odd dimension is half of that of the full superspace. All $N = 2$ theories mentioned above are naturally described by *Grassmann analytic* superfields, i.e., the unconstrained superfields in this subspace. A similar kind of analyticity underlies the $N = 3$ gauge theory [G5, G6].

A most unusual and novel feature of the analytic superfields is the unavoidable presence of infinite sets of auxiliary and/or gauge degrees of freedom in their component expansions. They naturally emerge from the harmonic expansions on the two-sphere S^2 with respect to a new sort of bosonic coordinates, the harmonic variables, which describe S^2 in a parametrization-independent way. These infinite sets, instead of being a handicap, proved to be very helpful indeed. It is due to their presence in the analytic superfield describing the $N = 2$ scalar multiplet (hypermultiplet) [F1, S12] off shell that one can circumvent the so-called 'no-go' theorem [H18, S21] claiming that such a formulation is not possible. In fact, the no-go theorems always implicitly assume the existence of a *finite* set of auxiliary fields.

The Grassmann analytic superfields with their infinite towers of components can be handled in much the same way as ordinary superfields, using a set of simple rules and tools. In [G14, G15] we worked out the quantization scheme for the $N = 2$ matter and gauge theories in harmonic superspace. The crucial importance of formulating quantum perturbation theory in supersymmetric models in terms of *unconstrained* off-shell superfields has repeatedly been pointed out in the literature (see, e.g., [H16]). Such formulations allow one to understand the origin of many remarkable properties of quantum supersymmetric theories which seem miraculous in the context of the component or constrained superfield formulations. Above all, this concerns the cancellation of ultraviolet divergences. Harmonic superspace is the only known approach which provides unconstrained off-shell formulations of both the matter and gauge $N = 2$ multiplets and as such it is indispensable for quantum calculations in the theories involving these multiplets. A particular representative of this class of theories is $N = 4$ super-Yang–Mills theory which, from the $N = 2$ perspective, is just the minimal coupling of the hypermultiplet in the adjoint representation of the gauge group to the $N = 2$ super-Yang–Mills multiplet.

It is worthwhile to emphasize that the harmonic superspace approach is very close to the twistor one which is an effective tool for solving the self-dual Yang–Mills and Einstein equations. In fact, harmonic superspace could be regarded as an isotwistor superspace. However, even when applied to the purely bosonic self-duality problems, the harmonic space approach has some advantages, one of them being as follows. We use harmonics (the fundamental isospin 1/2 spherical functions) as abstract global coordinates spanning the whole two-sphere. This is in contrast with, e.g., polar or stereographic coordinates which require two charts on the sphere. So, if one succeeds in solving a self-duality equation in terms of harmonics, there will be no need to attack the famous Riemann–Hilbert problem which is central in conventional twistor approaches. We also wish to stress that the harmonic (super)space formalism heavily uses the Cartan coset technique, transparent and familiar to many physicists.

A surge of interest in the harmonic superspace methods and, above all, in the methods for off-shell quantum calculations was mainly motivated by two remarkable developments in our understanding of supersymmetric field theories during the 1990s.

The first one stems from the seminal paper by Seiberg and Witten [S5] where it was suggested that $N = 2$ gauge theories are exactly solvable at the full quantum level under some reasonable hypotheses like S duality intimately related to extended supersymmetry [W17]. The study of the structure of the quantum low-energy effective actions of $N = 2$ gauge theories, in both the perturbative and non-perturbative sectors, is of great importance in this respect. The quantum harmonic superspace methods were successfully applied for this purpose, in particular for computing the holomorphic and non-holomorphic contributions to the effective action (see [B14, B15, B16, I8] and references therein).

The second source of interest is the famous Maldacena AdS/CFT conjecture [G42, M1, W16]. This is the idea that the quantum $N = 4$ super Yang–Mills theory in the limit of large number of colors and strong coupling is dual to the type IIB superstring on $AdS_5 \times S^5$ and contains the corresponding supergravity as a sub-sector of its Hilbert space. This conjecture greatly stimulated thorough analysis of the structure of this exceptional gauge theory from different points of view using different calculational means. The harmonic superspace methods, as was shown in several recent papers [E1, E3, E4, E5, H14], can drastically simplify the calculations and allow one to make far reaching predictions in $N = 4$ super-Yang–Mills theory.

All this justifies the need for a comprehensive introduction to the harmonic superspace approach. We hope that the present book will meet, at least partly, this quest. Here we do not discuss the latest developments but prefer to concentrate on the basics of the harmonic superspace method. Some developments are briefly addressed in the Conclusions. When reading this book one may find it helpful to consult the reviews and books mentioned above. We also point out that there are a few papers devoted to the mathematical aspects of harmonic superspace and, in particular, to a more rigorous definition of it, e.g., [H3, H10, H12, R6, S4]. We do not address these special issues in our rather elementary exposition.

1.2 Brief summary

The present book has been conceived as a pedagogical review of all the extended supersymmetric $N = 2$ theories and of $N = 3$ Yang–Mills theory in the framework of harmonic superspace. The details of these theories are discussed, as well as some applications. A special emphasis is put on their geometrical origin and on the relationship with hyper-Kähler and quaternionic complex manifolds which appear as the target manifolds of $N = 2$ supersymmetric sigma models in a flat background and in the presence of supergravity, respectively. The Cartan coset techniques are used systematically with emphasis on their power and simplicity. The self-duality Yang–Mills and Einstein equations are treated in this language with stress on their deep affinity with $N = 2$ supersymmetric theories and on comparing the harmonic space approach with the twistor one.

A detailed outline of the content of this book is given at the end of Chapter 1. In order to help the reader, we preface the main body of the book with an overview of the basic ideas, notions and origins. We begin with a discussion of spaces and superspaces for the realization of symmetries and supersymmetries, emphasizing the importance of making the right choice: *The same symmetry can be realized in different ways, one of them being much more appropriate for a given problem than the others.*

1.3 Spaces and superspaces

Manifestly invariant formulations of field theories make use of some space (or *superspace) where a given symmetry (or supersymmetry) is realized geometrically by coordinate transformations.* Two examples are well known:

(i) In Minkowski space $\mathbb{M}^4 = (x^a)$ the Poincaré group transformations have the form

$$x'^a = \Lambda^a_b x^b + c^a\,. \tag{1.1}$$

In classical and quantum field theories the action principle and the equations of motion are manifestly invariant under (1.1), the form of the corresponding field transformations being completely fixed by (1.1) and the tensor properties of the field, e.g.,

$$f'(x') = f(x) \tag{1.2}$$

for a scalar field $f(x)$. It is important that this transformation law does not depend on the model under consideration.

(ii) One usually attempts to formulate manifestly invariant N-extended supersymmetric theories in the standard superspace [S1]

$$\mathbb{R}^{4|4N} = (x^a, \theta^\alpha_i, \bar\theta^{\dot\alpha i})\,, \qquad i = 1, 2, \ldots, N \tag{1.3}$$

involving the spinor *anticommuting* coordinates $\theta^\alpha_i, \bar\theta^{\dot\alpha i}$ in addition to x^a. Their transformation rules under the Poincaré group are evident, while the transformations under supersymmetry (supertranslations with anticommuting parameters $\epsilon^\alpha_i, \bar\epsilon^{\dot\alpha i}$) are given by

$$\delta x^a = i(\epsilon^i\sigma^a\bar\theta_i - \theta^i\sigma^a\bar\epsilon_i)\,, \qquad \delta\theta^\alpha_i = \epsilon^\alpha_i\,, \qquad \delta\bar\theta^{\dot\alpha i} = \bar\epsilon^{\dot\alpha i}\,. \tag{1.4}$$

Superfields $\Phi(x, \theta, \bar\theta)$ are defined as functions on this superspace and their transformation law is completely determined by (1.2). For example, for a scalar superfield

$$\Phi'(x', \theta', \bar\theta') = \Phi(x, \theta, \bar\theta)\,. \tag{1.5}$$

Of course, this law is model-independent. Expanding a superfield $\Phi(x, \theta, \bar\theta)$ in powers of the spinor (anticommuting, hence nilpotent) variables $\theta, \bar\theta$ yields a finite set of usual fields $f(x), \psi^\alpha(x), \ldots$, called components of the superfield.

As an alternative to $\mathbb{R}^{4|4N}$, N-extended supersymmetry can also be realized in the so-called chiral superspace $\mathbb{C}^{4|2N}$ which is *complex* and contains only *half* of the spinor coordinates [F13]:

$$\delta x^a_L = -2i\theta^i_L\sigma^a\bar\epsilon_i\,, \qquad \delta\theta^\alpha_{L\,i} = \epsilon^\alpha_i\,. \tag{1.6}$$

In fact, the real superspace $\mathbb{R}^{4|4N}$ can be viewed as a real hypersurface in the complex superspace $\mathbb{C}^{4|2N}$:

$$x^a_L = x^a + i\theta^i\sigma^a\bar\theta_i\,, \qquad \theta^\alpha_{L\,i} = \theta^\alpha_i\,, \qquad \overline{\theta^\alpha_{L\,i}} = \bar\theta^{\dot\alpha\, i}\,. \tag{1.7}$$

1.4 Chirality as a kind of Grassmann analyticity

The superfields $\Phi(x_L, \theta_L) = \Phi(x + i\theta\sigma\bar{\theta}, \theta)$ defined in $\mathbb{C}^{4|2N}$ can be treated as Grassmann analytic superfields. Indeed, they obey the constraint

$$\bar{D}_{\dot{\alpha}i}\Phi = \left(-\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}i}} - i(\theta_i\sigma^a)_{\dot{\alpha}}\frac{\partial}{\partial x^a}\right)\Phi = 0\,, \tag{1.8}$$

where $\bar{D}^i_{\dot{\alpha}}$ is the covariant (i.e., commuting with the supersymmetry transformations) spinor derivative. In the basis $(x_L, \theta_L, \bar{\theta})$ this derivative simplifies to $\bar{D}^i_{\dot{\alpha}} = -\partial/\partial\bar{\theta}^{\dot{\alpha}i}$. Then the constraint (1.8) takes the form of a Cauchy–Riemann condition,

$$\frac{\partial\,\Phi}{\partial\,\bar{\theta}^{\dot{\alpha}}_i} = 0\,, \tag{1.9}$$

which means that Φ is a function of θ_L but is independent of $\bar{\theta}$ (cf. the standard theory of analytic functions where the Cauchy–Riemann condition $\partial\, f(z)/\partial\bar{z} = 0$ means that the function depends on the variable z and is independent of its conjugate $\bar{z}$). The notion of Grassmann analyticity [G8] in this simplest form is most useful in $N = 1$ supersymmetry. In this book the reader will see that there exist non-trivial generalizations of this concept which underlie the $N = 2$ and $N = 3$ supersymmetric theories.

It should by emphasized that finding the adequate superspace for a given theory is, as a rule, a non-trivial problem. The above superspaces $\mathbb{R}^{4|4N}$ and $\mathbb{C}^{4|2N}$ prove to be appropriate for off-shell formulations only in the simplest case of $N = 1$ supersymmetry. These 'standard' superspaces cease to be so useful in the extended ($N > 1$) supersymmetric theories. Finding and using the adequate superspaces for $N = 2,\ 3$ is the main subject of this book.

Now, before approaching the main problem, we recall in a few words some key points in $N = 1$ supersymmetry.

1.5 *N* = 1 chiral superfields

As already said, $N = 1$ supersymmetric theories can be formulated in the superspaces $\mathbb{R}^{4|4}$ or $\mathbb{C}^{4|2}$. Consider, for example, the simplest $N = 1$ supermultiplet, the matter one. On shell it contains a spin $1/2$ field ψ_α and a complex scalar field $A(x)$. In $\mathbb{R}^{4|4}$ it can be described by a scalar superfield $\Phi(x, \theta, \bar{\theta})$. However, the latter involves too many fields in its θ expansion: Four real scalars, two Majorana spinors and a vector of various dimensions. To eliminate the extra fields, it is necessary to impose a *constraint* on the superfield, which turns out to be just the chirality (Grassmann analyticity) condition

$$\bar{D}_{\dot{\alpha}}\Phi = 0\,. \tag{1.10}$$

As explained above, this constraint means that Φ is an analytic superfield. In the $N = 1$ case the expansion of such a superfield (written down in the chiral basis) is very short [F13]:

$$\Phi(x,\theta,\bar\theta) = \Phi(x_L,\theta_L) = \phi(x_L) + \theta_L^\alpha \psi_\alpha(x_L) + \theta_L^\alpha \theta_{L\alpha} F(x_L)\,. \qquad (1.11)$$

The fields ϕ, ψ_α, F form the *off-shell* $N = 1$ matter supermultiplet.

The chiral ($N = 1$ analytic) superspace $\mathbb{C}^{4|2}$ is the cornerstone of all the $N = 1$ theories: They are either formulated in terms of chiral superfields (matter and its self-couplings) or are based on gauge principles which respect chirality (Yang–Mills and supergravity and their couplings to matter). The reader will see that for $N = 2$ and $N = 3$ the suitably modified concept of Grassmann analyticity will also be crucial.

1.6 Auxiliary fields

Besides the physical fields $\phi(x_L)$, $\psi_\alpha(x_L)$, the superfield $\Phi(x_L,\theta_L)$ also contains an *auxiliary* complex scalar field $F(x_L)$ of non-physical dimension 2. As a consequence, this field can only appear in an action without derivatives and thus can be eliminated by its equation of motion. In the presence of auxiliary fields the supersymmetry transformations are *model-independent* and so have the same form off and on shell. They form a *closed* supersymmetry algebra. For example, in the case of the chiral scalar superfield above one obtains from (1.5), (1.6) and (1.11)

$$\begin{aligned} \delta\phi(x) &= -\epsilon^\alpha \psi_\alpha(x)\,, \\ \delta\psi_\alpha(x) &= -2i\sigma^a_{\alpha\dot\alpha}\bar\epsilon^{\dot\alpha}\ \partial_a\phi(x) - 2\epsilon_\alpha F(x)\,, \\ \delta F(x) &= -i\bar\epsilon^{\dot\alpha}\sigma^a_{\alpha\dot\alpha}\ \partial_a\psi^\alpha(x)\,. \end{aligned} \qquad (1.12)$$

The commutator of two such supertranslations yields an ordinary translation with a parameter composed in accordance with the supersymmetry algebra. (See Chapter 2 for more details on the realization of supersymmetry in terms of fields.)

Of course, one can find a realization of supersymmetry on the physical fields only, with the auxiliary fields eliminated by the equations of motion of a given model. In fact, the first known realizations of supersymmetry were of just such a kind, and it was to some extent an ‘art’ to simultaneously find the invariant action and the supersymmetry transformations leaving it invariant. In contrast with the transformations in the presence of auxiliary fields, now one has:

(i) Supersymmetry transformations depending on the choice of the specific field model. They are in general non-linear and the structure of this non-linearity varies from one action to another.

(ii) The algebra of these transformations closes only modulo the equations of motion, i.e., on shell. Such algebras are referred to as *open* or *soft*.

These complications cause difficulties when trying to exploit the consequences of supersymmetry, in particular, in studying the ultraviolet behavior. Working in a manifestly invariant manner, in terms of the appropriate superfields, has undeniable advantages for such purposes. Note that some people prefer to avoid the use of superfields and instead work directly with the off-shell supermultiplets of fields including the auxiliary ones (e.g., $\phi(x)$, $\psi_\alpha(x)$, $F(x)$ in our $N = 1$ example). Then one needs a set of rules for handling such multiplets, known as *tensor calculus*. The superfield approach automatically reproduces all such rules in a nice geometrical way. This concerns the composition rule for supermultiplets (it amounts to multiplication of superfields), the building of invariant actions, etc.

The reader should realize that the notion of auxiliary fields is not peculiar to supersymmetry, it also appears in the usual non-supersymmetric theories. For instance, the Coulomb field is auxiliary in quantum electrodynamics.

The auxiliary fields play an extremely important rôle in the theories with extended supersymmetry, their number there may even become infinite. The reader will learn from the present book that this is due to a new feature of the harmonic superspace: It involves auxiliary bosonic coordinates. This superspace of a new kind is the only one that provides us with a systematic tool for off-shell realizations of all the $N = 2$ extended supersymmetries and the $N = 3$ Yang–Mills theory.

1.7 Why standard superspace is not adequate for $N = 2$ supersymmetry

'Not adequate' means that in the framework of the standard superspaces $\mathbb{R}^{4|8}$ and $\mathbb{C}^{4|4}$ it is impossible to find off-shell actions for an unconstrained description of all the $N = 2$ supersymmetric theories. We illustrate this on the example of the Fayet–Sohnius matter hypermultiplet [F1, S12]. On shell this supermultiplet contains four scalar fields forming an $SU(2)$ doublet $f^i(x)$ and two isosinglet spinor fields $\psi^\alpha(x), \bar{\kappa}^{\dot{\alpha}}(x)$. To incorporate them as components of a standard superfield one has to use [S12] an isodoublet superfield $q^i(x, \theta, \bar{\theta})$ defined in $\mathbb{R}^{4|8}$. Due to the large number of spinor variables this superfield contains a lot of redundant field components in addition to the physical ones listed above. The extra fields are eliminated by imposing the constraint [S12]

$$D^{(i}_\alpha q^{j)} = \bar{D}^{(i}_{\dot{\alpha}} q^{j)} = 0\,, \tag{1.13}$$

where (ij) means symmetrization and D^i_α, $\bar{D}^i_{\dot{\alpha}}$ are the supercovariant spinor derivatives obeying the algebra

$$\{D^i_\alpha, \bar{D}_{\dot{\alpha}j}\} = -2i\delta^i_j \sigma^a_{\alpha\dot{\alpha}} \frac{\partial}{\partial x^a} \tag{1.14}$$

(for their precise definition see Chapter 3). These constraints eliminate the extra field components of q^i, leaving only the above physical fields (and their derivatives in the higher terms of the θ expansion):

$$q^i(x,\theta,\bar\theta) = f^i(x) + \theta^{i\alpha}\psi_\alpha(x) + \bar\theta^i_{\dot\alpha}\bar\kappa^{\dot\alpha}(x) + \text{ derivative terms.} \tag{1.15}$$

However, at the same time the above constraints put all the physical fields on the free mass shell:

$$\Box f^i(x) = (\sigma^a)^{\alpha\dot\alpha}\frac{\partial}{\partial x^a}\psi_\alpha(x) = \sigma^a_{\alpha\dot\alpha}\frac{\partial}{\partial x^a}\bar\kappa^{\dot\alpha}(x) = 0\,. \tag{1.16}$$

The reason for this is that the constraints (1.13) are not integrable off shell: The supercovariant spinor derivatives do not anticommute. Equations (1.16) follow from the constraints (1.13) and the algebra (1.14), taking into account the definitions

$$f^i(x) = q^i|_{\theta=\bar\theta=0}\,, \qquad \psi_\alpha(x) = \frac{1}{2}D^i_\alpha q_i|_{\theta=\bar\theta=0}\,, \qquad \kappa^{\dot\alpha(x)} = \frac{1}{2}\bar D^{\dot\alpha}_i q^i|_{\theta=\bar\theta=0}\,. \tag{1.17}$$

In order to extend this theory off shell and to introduce interactions it has been proposed to relax, in one way or another, the constraints (1.13) [H15, Y1]. However, according to the general no-go theorem [H18, S21] (see Chapter 2), this is impossible in the framework of the standard $N = 2$ superspaces $\mathbb{R}^{4|8}$ or $\mathbb{C}^{4|4}$ using a *finite number* of auxiliary fields (or, equivalently, a *finite number* of standard $N = 2$ superfields). A natural way out was to look for other superspaces.

1.8 Search for conceivable superspaces (spaces)

Above we saw that it is helpful to consider different superspaces even in the simplest case $N = 1$. For any (super)symmetry there exists a number of admissible (super)spaces. The inadequacy of the standard superspaces $\mathbb{R}^{4|8}$ and $\mathbb{C}^{4|4}$ for off-shell realizations of $N = 2$ supersymmetry suggested to start searching through the list of other available superspaces. This list is provided by the standard coset construction due to E. Cartan [C4]* that allows one to classify the different (super)spaces of some (super)group G and to handle them effectively. One has to examine the conceivable quotients (we prefer the term 'coset') G/H of the group G over some of its subgroups H. For instance, Minkowski space is the coset $\mathbb{M}^4 = \mathcal{P}/\mathcal{L} = (x^a)$ of the Poincaré group $\mathcal{P}$ over its Lorentz subgroup $\mathcal{L}$. As we shall see later, the Poincaré group for the Euclidean space $\mathbb{R}^4$ can also be realized in another way, using the coset space $\mathcal{P}/SU(2)\times U(1)$, with $SU(2)\times U(1)$ being a subgroup of the rotation group $SO(4) = SU(2)\times SU(2)$. This space is closely related to the so-called *twistor space* (more precisely, the traditional twistor space is related by a similar procedure to the Poincaré group of the complexified Minkowski space $\mathbb{M}^4$).

* Subsequently rediscovered by physicists [C11, O3, V3].

Analogously, the standard real superspaces $\mathbb{R}^{4|4N}$ are the coset spaces

$$\mathbb{R}^{4|4N} = \frac{Su\mathcal{P}_N}{\mathcal{L}} = (x^a, \theta_i^\alpha, \bar{\theta}^{\dot{\alpha}i}) , \qquad (1.18)$$

where $Su\mathcal{P}_N$ is the N-extended super-Poincaré group involving the generators of the Poincaré group and the spinor supersymmetry generators Q_α^i, $\bar{Q}_{\dot{\alpha}i}$. In the same way, the chiral superspaces are the following coset spaces

$$\mathbb{C}^{4|2N} = \frac{Su\mathcal{P}_N}{\{\mathcal{L}, \bar{Q}_{\dot{\alpha}i}\}} = (x^a, \theta_i^\alpha) . \qquad (1.19)$$

Note the important difference between (1.18) and (1.19). In the latter the stability *supergroup* contains half of the spinor generators in addition to the Lorentz group ones. In Chapter 3 the coset techniques [C4, C11, O3, V3] are presented in detail. These techniques provide simple rules on how to find explicit transformation laws, how to construct invariants making use of covariant derivatives (obtained from the appropriate Cartan forms), etc.

1.9 *N* = 2 harmonic superspace

Certainly, $\mathbb{R}^{4|4N}$ and $\mathbb{C}^{4|2N}$ do not exhaust the list of possible superspaces for realizations of N-extended supersymmetry. Let us briefly outline some general features of $N = 2$ harmonic superspace, our main topic of interest in this book.

The $N = 2$ superalgebra

$$\left\{Q_\alpha^i, \bar{Q}_{\dot{\alpha}j}\right\} = 2\delta_j^i(\sigma^a)_{\alpha\dot{\alpha}} P_a , \qquad i, j = 1, 2 \qquad (1.20)$$

possesses an $SU(2)$ group of automorphisms, Q_α^i, $\bar{Q}_{\dot{\alpha},j}$ being $SU(2)$ doublets (indices i, j) and P_a being a singlet. In the standard case of eqs. (1.18) and (1.19) (with $N = 2$) this $SU(2)$ can be viewed as present both in the numerator and the denominator, thus effectively dropping out. To obtain the harmonic superspace, one has to keep only the $U(1)$ subgroup of $SU(2)$ in the denominator instead of the whole $SU(2)$:

$$\mathbb{H}^{4+2|8} = \frac{Su\mathcal{P}_2}{\mathcal{L}} \times \frac{SU(2)}{U(1)} . \qquad (1.21)$$

In other words, one has to enlarge the $N = 2$ supersymmetry group by its automorphisms group $SU(2)$ realized in the coset space $SU(2)/U(1)$. The latter is a two-dimensional space known to have the topology of the two-sphere S^2. So, harmonic superspace is a tensor product of $\mathbb{R}^{4|8}$ and a two-sphere S^2.

1.10 Dealing with the sphere S^2

Before discussing the harmonic superspace as a whole it is instructive to study its much more familiar part $SU(2)/U(1)$. Of course, one could choose polar

(θ, ϕ) or stereographic $(t, \bar{t})$ coordinates on this sphere. However, it turns out much more convenient to coordinatize it by some 'zweibeins' $u^{+i}, u_i^- = \overline{u^{+i}}$ having $SU(2)$ indices i and $U(1)$ charges $\pm$. After imposing the constraint

$$u^{+i} u_i^- = 1\,, \tag{1.22}$$

the matrix

$$\| u \| = \begin{pmatrix} u_1^+ & u_1^- \\ u_2^+ & u_2^- \end{pmatrix} = \frac{1}{\sqrt{1 + t\bar{t}}} \begin{pmatrix} e^{i\psi} & -\bar{t}e^{-i\psi} \\ te^{i\psi} & e^{-i\psi} \end{pmatrix}, \qquad 0 \le \psi < 2\pi \tag{1.23}$$

represents the group $SU(2)$ in the familiar stereographic parametrization. We are interested in its coset space $SU(2)/U(1)$. This means that the zweibeins have to be defined up to a $U(1)$ phase corresponding to a transformation of the $U(1)$ group in the coset denominator:

$$u_i^{+'} = e^{i\alpha} u_i^+\,, \qquad u_i^{-'} = e^{-i\alpha} u_i^- \tag{1.24}$$

(this transformation can be realized as right multiplications of the matrix (1.23) with the Pauli matrix τ^3 as the generator). So the phase ψ in the parametrization (1.23) is inessential and one effectively deals only with the complex coordinates $t, \bar{t}$. In order for the phase not to show up, the 'functions' on the sphere must have a *definite* $U(1)$ charge q and, as a consequence, all the terms in their harmonic expansion must contain only products of zweibeins u^+, u^- of the given charge q. For instance, for $q = +1$

$$f^+(u) = f^i u_i^+ + f^{(ijk)} u_i^+ u_j^+ u_k^- + \cdots\,. \tag{1.25}$$

Such quantities undergo homogeneous $U(1)$ phase transformations, according to their overall charge. This requirement on the harmonic functions can be called $U(1)$ charge preservation. In each term in (1.25) complete symmetrization in the indices $i, j, k, \ldots$ is assumed, otherwise the term can be reduced to the preceding ones by eq. (1.22).

In fact, the zweibeins u_i^+, u_i^- are the fundamental spin 1/2 spherical harmonics familiar from quantum mechanics, and (1.25) is an example of a harmonic decomposition on S^2. This is why we call u_i^+, u_i^- *harmonic variables* (or simply 'harmonics').

1.10.1 Comparison with the standard harmonic analysis

We would like to point out the following important features of the harmonic space approach that differ from the standard ones in textbooks and reviews on harmonic analysis [B9, C2, C3, G31, G37, H5, H6, V2, W13]:

(i) We use the harmonics themselves as coordinates of the sphere. This amounts to refraining from using any explicit parametrization like the stereographic one (1.23). Instead, we assume the defining constraint (1.22) together with the requirement of $U(1)$ charge preservation.

(ii) We deal with symmetrized products of harmonics instead of sets of special functions, like the Jacobi polynomials or the spherical functions familiar from the harmonic analysis on the two-dimensional sphere.

These formal modifications turn out very convenient for the following main reasons:

(i) The coefficients in the harmonic expansions (like f^i, $f^{(ijk)}, \ldots$ in (1.25)) transform as irreducible representations of the $SU(2)$ group of the coset numerator. This is of special value in $N = 2$ supersymmetry because the $N = 2$ supermultiplets are classified, in particular, according to the $SU(2)$ automorphism group.

(ii) Working with local coordinates one is confronted with the Riemann–Hilbert problem: Two maps are needed to cover the two-sphere or the extended complex plane. So, given a function which is well defined in the northern hemisphere, one has to worry about defining it consistently in the southern hemisphere. Remarkably enough, this problem does not appear if one exploits the harmonics u^+_i, u^-_i as 'global' coordinates on S^2. If one has succeeded in solving some equation in terms of harmonics, then the solution obtained is well defined on the entire sphere, after substitution of the parametrization (1.23) (or any other local one).

The latter statement can be illustrated by the following simple example. On the sphere S^2 one may introduce two covariant derivatives consistent with the constraint (1.22) and having $U(1)$ charges $+2$ and -2:

$$D^{++} = u^{+i}\frac{\partial}{\partial u^{-i}} \qquad \text{and} \qquad D^{--} = u^{-i}\frac{\partial}{\partial u^{+i}}\,. \tag{1.26}$$

They will be heavily used in what follows and referred to as harmonic derivatives. In the twistor literature [I2, H2, H22, K8] these derivatives are known as the *edth* and *antiedth* operators and their expressions in terms of polar or stereographic coordinates are used, e.g.,

$$D^{++} f^{(q)}(u) = -e^{(q+2)i\psi}\left[(1+t\bar{t})\frac{\partial \phi^{(q)}}{\partial \bar{t}} + \frac{qt}{2}\phi^{(q)}\right]. \tag{1.27}$$

Here $f^{(q)}(u) = e^{iq\psi}\phi^{(q)}(t,\bar{t})$ is a harmonic function of $U(1)$ charge q. As explained above, it depends on the coordinate ψ associated with the $U(1)$ charge through a simple phase factor. In contrast to the harmonic form (1.26), the edth operator D^{++} (1.27) explicitly involves the $U(1)$ charge q.

From the definition (1.26) follow the obvious rules for the action of D^{++} on the harmonics

$$D^{++}u_i^+ = 0\,, \qquad D^{++}u_i^- = u_i^+\,. \tag{1.28}$$

Let us consider the simple harmonic differential equation

$$D^{++} f^+ = 0\,. \tag{1.29}$$

In harmonics its solution is immediately obtained from (1.25):

$$f^+ = f^i u_i^+\,, \tag{1.30}$$

where f^i are arbitrary constants. Indeed, f^+ has this form because all other terms in its harmonic expansion include u^-.

Now it is instructive to compare this 'harmonic' procedure with solving the same equation as a partial differential equation with respect to the complex coordinates $t, \bar{t}$. It is easy to find the general solution of this equation for $q = 1$ in the form

$$f^+(t, \bar{t}, \psi) = e^{i\psi}(1 + t\bar{t})^{-\frac{1}{2}} F(t)\,, \tag{1.31}$$

where $F(t)$ is an arbitrary holomorphic function. However, we are interested in solutions well-behaved on the whole two-sphere (we wish to solve the Riemann–Hilbert problem). This requirement restricts the function $F(t)$ to the form of a polynomial of degree 1: $F(t) = f^1 + f^2 t$, where $f^1,\ f^2$ are arbitrary constants. In this way one obtains the same solution (1.30) in the particular parametrization (1.23). Note, however, that the solution (1.30) is *manifestly* $SU(2)$ *covariant* (the constants f^i form a doublet) whereas in a particular parametrization $SU(2)$ is realized as a non-linear coordinate transformation.

Finally, a word about integration on the two-sphere. In the harmonic approach it is defined by the following formal rules:

$$\int du\ 1 = 1\,, \qquad \int du\ u^+_{(i_1}..u^+_{i_k}u^-_{i_{k+1}}..u^-_{i_{k+l})} = 0\,. \tag{1.32}$$

This definition means the vanishing of the integrals of any spherical function with spin (represented by symmetrized products of harmonics). Of course, it admits integration by parts, etc. These rules can be justified by the use of some specific parametrization for the harmonics, e.g., (1.23). However, the abstract form (1.32) is most convenient in field theory, as the reader will have a number opportunities to see.

1.11 Why harmonic superspace helps

We now return to the harmonic superspace $\mathbb{H}^{4+2|8} = \mathbb{R}^{4|8} \times SU(2)/U(1)$ with the coordinates $\{x^a, \theta_i^\alpha, \bar{\theta}^{\dot{\alpha} i}, u_i^\pm\}$. We explain, on the example of the Fayet–Sohnius hypermultiplet, why adding the two-sphere is so crucial for the off-shell formulation of $N = 2$ supersymmetric field theories.

With the help of the harmonics $u_i^{\pm}$ we can give the constraints (1.13) another, more suggestive form. Let us multiply them by u_i^+, u_j^+. Introducing the notation

$$D_\alpha^+ = u_i^+ D_\alpha^i, \qquad \bar{D}_{\dot\alpha}^+ = u_i^+ \bar{D}_{\dot\alpha}^i \tag{1.33}$$

and

$$q^+ = u_i^+ q^i \tag{1.34}$$

one rewrites (1.13) as

$$D_\alpha^+ q^+ = 0\,, \qquad \bar{D}_{\dot\alpha}^+ q^+ = 0\,. \tag{1.35}$$

Equation (1.34) simply means that q^+ depends linearly on the harmonics u_i^+ (in this basis). It can be recast in an equivalent form using the harmonic derivative D^{++}:

$$D^{++} q^+ = 0 \tag{1.36}$$

(cf. (1.29)). Equations (1.35) and (1.36) together are clearly equivalent to the constraints (1.13). However, these equations turn out to have a deeper meaning than (1.13). First of all, we remark that the derivatives entering the modified constraints (1.35), (1.36) mutually (anti)commute,

$$\{D_\alpha^+, \bar{D}_{\dot\alpha}^+\} = [D^{++}, D_\alpha^+] = [D^{++}, \bar{D}_{\dot\alpha}^+] = 0\,, \tag{1.37}$$

in contrast to the derivatives entering the original constraints (1.13). This property is of great significance. Owing to it one can consider equations (1.35) as the *generalized Cauchy–Riemann condition of Grassmann analyticity*. To reveal its meaning one should choose an adequate basis in superspace, the so-called *analytic basis* (an analog of the chiral basis (1.7)):

$$x_A^a = x^a - 2i\theta^{(i}\sigma^a\bar{\theta}^{j)}u_i^+ u_j^-\,, \qquad \theta_{A\alpha}^{\pm} = u_i^{\pm}\theta_\alpha^i\,, \qquad \bar{\theta}_{A\dot\alpha}^{\pm} = u_i^{\pm}\bar{\theta}_{\dot\alpha}^i\,. \tag{1.38}$$

In this basis the spinor derivatives D_α^+ and $\bar{D}_{\dot\alpha}^+$ become simple partial derivatives and the constraints (1.35) take the form

$$D_\alpha^+ q^+ = \frac{\partial}{\partial\theta^{-\alpha}} q^+ = 0\,, \qquad \bar{D}_{\dot\alpha}^+ q^+ = \frac{\partial}{\partial\bar{\theta}^{-\dot\alpha}} q^+ = 0\,. \tag{1.39}$$

Like the Cauchy–Riemann condition of ordinary analyticity or that of $N = 1$ Grassmann analyticity (chirality), equations (1.39) express the fact that q^+ is independent of half of the relevant variables, this time of the spinor coordinates $\theta^{-\alpha}, \bar{\theta}^{-\dot\alpha}$. Their solution is

$$q^+ = q^+(x_A, \theta^+, \bar{\theta}^+, u^{\pm})\,. \tag{1.40}$$

The same condition can be imposed on harmonic superfields with $U(1)$ charges different from $+1$. We refer to conditions like (1.35) or (1.39) as *Grassmann analyticity* conditions and to the subspace

$$\mathbb{HR}^{4+2|8} = (x_A, \theta^+, \bar{\theta}^+, u^{\pm}) = (\zeta, u^{\pm}) \tag{1.41}$$

as *Grassmann analytic superspace*. It contains only half of the original spinor coordinates (those having $U(1)$ charge equal to $+1$) and yet it is closed under the full $N = 2$ supersymmetry transformations. We can state that all the $N = 2$ supersymmetric theories (matter, Yang–Mills and supergravity) are most adequately formulated in its framework.

In the analytic basis (1.38) the harmonic derivative takes the form

$$D^{++} = u^{+i}\frac{\partial}{\partial u^{-i}} - 2i\theta^{+}\sigma^{a}\bar{\theta}^{+}\frac{\partial}{\partial x_A^a}\,, \tag{1.42}$$

where one sees space-time derivatives. As a consequence, eq. (1.36) becomes dynamical and yields the free equations of motion for all physical matter fields. So, we have succeeded in reformulating the original constraints (1.13) (whose rôle was to eliminate the extra fields and simultaneously to put the remaining physical fields on shell) into the analyticity constraints (1.39) (having the evident solution (1.40)) and the (free) equation of motion (1.36). This was achieved due to the presence of the harmonics $u_i^{\pm}$. Now we can go a step further and introduce general self-interactions. This simply amounts to inserting a general source J^{+++} of $U(1)$ charge $+3$ in the right-hand side of eq. (1.36):

$$D^{++}q^{+} = J^{+++}(q^{+}, u^{\pm})\,. \tag{1.43}$$

One has to realize that the harmonic expansion of the analytic superfield q^{+} contains an *infinite number* of auxiliary fields. This is how harmonic superspace gets around the no-go theorem [H18, S21] asserting that it is not possible to describe the above complex hypermultiplet off shell with a finite number of auxiliary fields.

1.12 $N = 2$ supersymmetric theories

Now we are ready to very briefly overview the $N = 2$ matter, Yang–Mills and supergravity theories formulated in harmonic superspace in order to give some guidelines to the main text where the reader will find all the details and, the authors hope, a deeper insight.

1.12.1 $N = 2$ matter hypermultiplet

The general action for $N = 2$ supersymmetric matter is written down as an analytic superspace integral:

$$S = -\int du\, d\zeta^{(-4)}\, [\tilde{q}^{+}D^{++}q^{+} - L^{+4}(q^{+}, \tilde{q}^{+}, u)]\,. \tag{1.44}$$

Here L^{+4} is an arbitrary function of its arguments carrying $U(1)$ charge $+4$. It gives rise to the source term in the equation of motion (1.43), $J^{+++} =$

$\partial L^{+4}/\partial\tilde{q}^{+}$. The operation $\tilde{}$ is a special involution preserving the analytic harmonic superspace (1.41) (it is reduced to ordinary complex conjugation for the u-independent quantities), and the integration measure of the analytic superspace is defined as

$$d\zeta^{(-4)} = d^4x\, d^2\theta^+\, d^2\bar{\theta}^+ \,. \tag{1.45}$$

This measure carries negative $U(1)$ charge because Grassmann integration is equivalent to differentiation [B7] with respect to the odd coordinates of the analytic superspace $\theta^+_\alpha, \bar{\theta}^+_{\dot\alpha}$. These formulas look simple. However, in order to be able to effectively work with them one needs precise definitions and details, especially of the harmonic calculus on S^2. In particular, one needs to know how to solve differential equations on S^2 to which the auxiliary field equations of motion following from (1.44) are reduced. All this will be explained in the main body of the book.

Here we make a few comments only. The off-shell action (1.44) corresponds to the general $N = 2$ supersymmetric sigma model. The target spaces of such sigma models are known to belong to a remarkable class of $4n$-dimensional complex manifolds: According to the theorem of ref. [A2] they are the so-called hyper-Kähler manifolds. This means that they admit a triplet of covariantly constant complex structures forming the algebra of quaternionic units or, equivalently, that their holonomy group lies in $Sp(n)$. The essentially new point in the harmonic approach is that the interaction Lagrangian $L^{+4}(q, \tilde{q}, u)$ appears as the hyper-Kähler potential which encodes the complete information about the local properties of a given manifold. For example, $L^{+4} = \lambda(q^+)^2(\tilde{q}^+)^2$ describes the well-known Taub–NUT hyper-Kähler manifold. It is worthwhile mentioning that the four-dimensional hyper-Kähler manifolds (corresponding to a single hypermultiplet action) represent solutions of the self-dual Einstein equations, among them the gravitational instantons.

1.12.2 $N = 2$ Yang–Mills theory

$N = 2$ supersymmetric Yang–Mills theory is similar to ordinary ($N = 0$) Yang–Mills theory. It is based on making an internal symmetry group local in the analytic harmonic superspace $(x_A, \theta^+, \bar{\theta}^+, u^\pm) = (\zeta, u^\pm)$ (1.41) (instead of just Minkowski space in the $N = 0$ case):

$$\delta q^+_r = i\lambda^k (t_k)_{rs} q^+_s \quad \Rightarrow \quad \delta q^+_r(\zeta, u^\pm) = i\lambda^k(\zeta, u^\pm)(t_k)_{rs} q^+_s(\zeta, u^\pm)\,, \tag{1.46}$$

where t_k are the generators of the internal symmetry group and λ^k are the corresponding parameters. As usual, one should covariantize the derivatives entering the action. In our case, it is the harmonic one:

$$D^{++} \quad \Rightarrow \quad \mathcal{D}^{++} = D^{++} + iV^{++}(\zeta, u)\,. \tag{1.47}$$

The gauge connection $V^{++}(\zeta, u) = V^{++k}(x_A, \theta^+, \bar{\theta}^+, u^{\pm})t_k$ is a Lie algebra-valued analytic harmonic superfield. It transforms under the gauge group according to the standard rule

$$\delta V^{++}(\zeta, u) = -\mathcal{D}^{++}\lambda(\zeta, u)\,, \qquad \lambda(\zeta, u) \equiv \lambda^k(\zeta, u)t_k\,. \tag{1.48}$$

This superfield describes just the off-shell $N = 2$ Yang–Mills supermultiplet, as the reader will see in Chapter 7. This multiplet consists of a gauge vector field $A_a(x)$, a doublet of Weyl spinors $\psi^i_\alpha(x)$, a complex scalar field $\phi(x)$ and a triplet of auxiliary fields $D^{(ij)}(x)$. It should be pointed out that, as opposed to the matter hypermultiplet $q^+(\zeta, u)$, the gauge superfield $V^{++}(\zeta, u)$ contains a *finite* number of auxiliary fields. Instead, it has an infinite number of pure gauge degrees of freedom which are gauged away by the transformations (1.48). The harmonic superspace formulation reveals the close similarity between $N = 2$ super-Yang–Mills theory and the ordinary bosonic ($N = 0$) Yang–Mills theory.

Having defined the covariant derivative (1.47), one can immediately introduce the minimal Yang–Mills–matter coupling by simply covariantizing the action (1.44). The details of how to construct an invariant action for the Yang–Mills superfield itself will be given in Chapter 7. The general class of $N = 2$ Yang–Mills field theories in interaction with hypermultiplets is known to contain a subclass of four-dimensional *ultraviolet finite quantum field theories* (in particular, $N = 4$ Yang–Mills theory). They also reveal remarkable properties of duality [S5]. Harmonic superspace considerably simplifies many aspects and makes manifest many features of these theories, e.g., the proof of non-renormalization theorems, finding out the full structure of the quantum effective actions, etc.

The harmonic approach is also convenient for the description of general non-minimal self-couplings of vector $N = 2$ supermultiplets. These theories are unique because they are the only $N = 2$ supersymmetric field-theoretical models that admit a natural chiral structure of interactions. For this reason they may be useful in the phenomenological context as a possible basis of $N = 2$ GUT models. Sigma models inherent to these couplings are of interest in their own right. Their tangent manifolds are of some special Kähler type [C7, C10] and have been discussed in connection with the so-called $c*$-map [C8, C9].

A historical comment is due here. Unlike $N = 2$ matter, the $N = 2$ Yang–Mills theory can be formulated in terms of standard *unconstrained* $\mathbb{R}^{4|8}$ superfields (since it only involves a finite set of auxiliary fields). Such a more 'traditional' formulation of $N = 2$ Maxwell theory was first given in [M3] and its non-Abelian version in [G28]. The main drawback of this approach is the lack of geometric meaning of the Yang–Mills prepotential and gauge group, which makes quantization particularly cumbersome. In Chapter 7 we shall show that these $\mathbb{R}^{4|8}$ objects can be derived from the harmonic superspace ones by a special choice of gauge with respect to the transformations (1.46), (1.48).

1.12.3 $N = 2$ supergravity

Now we make a few comments on the $N = 2$ supergravity theory. It is an extension of Einstein's theory of gravity describing the metric field $g_{mn}(x)$ (graviton) and its $N = 2$ superpartners: an $SU(2)$ doublet of Rarita–Schwinger fields $\psi^i_{m\alpha}(x)$, $\bar{\psi}^i_{m\dot{\alpha}}(x)$ ('gravitini') and a vector gauge field $A_m(x)$ ('graviphoton'). The underlying principle is gauge invariance under some supergroup containing the diffeomorphism group of four-dimensional space-time as a subgroup. To formulate $N = 2$ supergravity theory one has to answer the following questions:

(i) What kind of superspace is appropriate?

(ii) What is the gauge supergroup needed?

(iii) What are the unconstrained prepotentials?

(iv) How to construct the invariant action?

(v) How many versions of the theory do exist and what are the differences between them?

The answers to the above questions given in this book are as follows:

(i) The superspace for $N = 2$ supergravity is harmonic superspace.

(ii) The appropriate $N = 2$ (conformal) supergravity gauge supergroup is the superdiffeomorphism group of the harmonic analytic superspace (ζ, u):

$$\begin{aligned} \delta x^m &= \lambda^m(\zeta, u)\,, \\ \delta\theta^{\mu+} &= \lambda^{\mu+}(\zeta, u)\,, \qquad \delta\bar{\theta}^{\dot{\mu}+} = \bar{\lambda}^{\dot{\mu}+}(\zeta, u)\,, \\ \delta u^+_i &= \lambda^{++}(\zeta, u)u^-_i\,, \qquad \delta u^-_i = 0\,, \end{aligned} \tag{1.49}$$

where the local parameters λ are arbitrary analytic harmonic functions. Note that only the harmonics u^+ but not u^- transform, a peculiarity due to the special realization of the $N = 2$ superconformal group (see Chapter 9).

(iii) As in $N = 2$ Yang–Mills theory, the $N = 2$ supergravity prepotentials appear in the covariantized harmonic derivative

$$\mathcal{D}^{++} = u^+_i \frac{\partial}{\partial u^-_i} + H^{++++} u^-_i \frac{\partial}{\partial u^+_i} + H^{++m} \frac{\partial}{\partial x^m_A} + H^{++\hat{\mu}+} \frac{\partial}{\partial \theta^{\hat{\mu}+}} \tag{1.50}$$

(here $\hat{\mu} = \mu, \dot{\mu}$). Covariantization is achieved by adding to the flat harmonic derivative $D^{++} = u^+_i \partial/\partial u^-_i$ appropriate vielbein terms with analytic vielbeins $H^{++++}(\zeta, u)$, $H^{++m}(\zeta, u)$, $H^{++\hat{\mu}+}(\zeta, u)$ (the counterparts of the gauge connection $V^{++}(\zeta, u)$ in the Yang–Mills case). These vielbeins are the unconstrained prepotentials of $N = 2$ supergravity. Their gauge transformation laws follow from (1.49).

In fact, the supergroup (1.49) and the prepotentials (1.50) are relevant to the so-called conformal (or Weyl) supergravity. This kind of supergravity possesses a somewhat bigger gauge symmetry than Einstein $N = 2$ supergravity. The extra gauge transformations have to be compensated by coupling conformal supergravity to some matter supermultiplets called *compensators*. This procedure is widely used in gravity and supergravity theories [D17, D18, D19, D20, D21, D22, F14, G26, G27].

(iv) The action for $N = 2$ Einstein supergravity is written down as the action for the $N = 2$ compensators in the background of $N = 2$ conformal supergravity. Two such compensating supermultiplets are needed. One of them is always an Abelian vector supermultiplet, but there exist several alternative choices for the second one.

(v) It should be stressed that different sets of compensators lead to different off-shell versions of $N = 2$ Einstein supergravity having different sets of auxiliary fields. In the harmonic superspace approach one can reproduce all the versions previously found in the component field approach [D15, D16, D17, D18, D19, F15]. Naturally, the latter always contains a *finite* set of auxiliary fields. Consequently, the corresponding compensators are described by *constrained* Grassmann analytic superfields (e.g., by the so-called tensor or non-linear multiplets). The presence of such constraints restricts the possible form of matter couplings, the latter have to be consistent with the former. For instance, the matter hypermultiplets self-couplings must possess some isometries. However, the harmonic superspace approach provides a new, 'principal' version of $N = 2$ Einstein supergravity with an *unconstrained* hypermultiplet q^+ as a compensator. This version admits the most general matter couplings. At the same time, it naturally contains an infinite number of auxiliary fields and thus could not be discovered by traditional methods. The bosonic target manifolds of the corresponding $N = 2$ sigma models are *quaternionic* [B1, B4] in contrast to the hyper-Kähler ones in the flat $N = 2$ case. The harmonic superspace approach clearly exhibits this important property [G7, G19] and offers an efficient tool for the explicit calculation of quaternionic metrics [G7].

1.13 $N = 3$ Yang–Mills theory

The harmonic superspace concept is not limited to $N = 2$ supersymmetry only. However, going to $N > 2$ requires some major changes. At present, $N = 3$ Yang–Mills theory is fully understood [G5, G6, G12]. Here are some of the basic ideas. $N = 3$ harmonic superspace is a tensor product of the standard real superspace $\mathbb{R}^{4|12}$ and the six-dimensional coset space $SU(3)/U(1) \times U(1)$, where $SU(3)$ is the automorphism group of $N = 3$ supersymmetry. So, instead of just one we now deal with *two* $U(1)$ charges. The corresponding harmonics

$$u^I_i = (u_i^{(1,0)}, u_i^{(0,-1)}, u_i^{(-1,1)}) \; ; \qquad u^i_I = \overline{u^I_i} \; ; \qquad i = 1,2,3 \qquad (1.51)$$

are subject to the defining conditions

$$u^i_I u^J_i = \delta^J_I \; ; \qquad u^i_I u^I_j = \delta^i_j \; ; \qquad \det u = 1 \, .$$

The spinor variables $\theta_{i\alpha}, \bar\theta^i_{\dot\alpha}$ form the representations 3 and $\underline{3}$ of $SU(3)$. With the help of the harmonics (1.51) they are projected onto six independent variables:

$$\theta_\alpha^{(-1,0)}\, , \quad \theta_\alpha^{(0,1)}\, , \quad \theta_\alpha^{(1,-1)}\, , \quad \bar\theta_{\dot\alpha}^{(1,0)}\, , \quad \bar\theta_{\dot\alpha}^{(0,-1)}\, , \quad \bar\theta_{\dot\alpha}^{(-1,1)}\, .$$

The analytic $N=3$ superspace contains only four of them, $\theta_\alpha^{(1,-1)}$, $\theta_\alpha^{(0,1)}$, $\bar\theta_{\dot\alpha}^{(1,0)}$, $\bar\theta_{\dot\alpha}^{(-1,1)}$ (and not half, as was the case in $N = 2$). The analytic Yang–Mills prepotentials $V^{(1,1)}, V^{(2,-1)}, V^{(-1,2)}$ are introduced as the gauge connections for the harmonic derivatives $\mathcal{D}^{(1,1)}, \mathcal{D}^{(2,-1)}, \mathcal{D}^{(-1,2)}$. They have the usual transformation law $\delta V^{(a,b)} = \mathcal{D}^{(a,b)}\lambda$, where λ is a chargeless analytic superfield parameter. The action is very unusual, it is written down as a Chern–Simons term:

$$\begin{aligned} S^{N=3}_{\rm SYM} \;=\; & \int du\, d\zeta_A^{(-2,-2)} {\rm Tr}\big(V^{(2,-1)}F^{(0,3)} + V^{(-1,2)}F^{(3,0)} + V^{(1,1)}F^{(1,1)} \\ & - i V^{(1,1)}[V^{(2,-1)}, V^{(-1,2)}]\big)\, , \qquad (1.52) \end{aligned}$$

where the three F's are the field strengths, e.g., $F^{(3,0)} = -i[\mathcal{D}^{(1,1)}, \mathcal{D}^{(2,-1)}]$. Note that the Chern–Simons-type action (1.52) was proposed as early as 1985 and it describes a very non-trivial dynamics. Nowadays Chern–Simons actions are becoming popular in connection with string field theory and topological field theory [W15].

There remain a lot of important problems in supersymmetric theories that one can hope to solve within the harmonic superspace approach. These techniques have already been employed to approach $N = 4$ supersymmetric Yang–Mills theory [S18], ten-dimensional Yang–Mills and supergravity theories in the context of superparticle and superstring models [G3, N3, N4, N5, S16, S17], etc.

1.14 Harmonics and twistors. Self-duality equations

The harmonic superspace approach has a close relationship to the famous twistor theory [P2, P3, P4]. Common for both is an extension of space-time (in twistor theory) and superspace (in the harmonic superspace approach) by adding some two-dimensional sphere S^2. In such an extended space the self-dual Yang–Mills