We are here led to considerations belonging to the Geometry of Position, a subject which, though its importance was pointed out by Leibnitz and illustrated by Gauss, has been little studied.

James Clerk Maxwell, *A Treatise on Electricity and Magnetism*, 1891

Introduction

The title of this book makes clear that we are after connections between electromagnetics, computation and topology. However, connections between these three fields can mean different things to different people. For a modern engineer, computational electromagnetics is a well-defined term and topology seems to be a novel aspect. To this modern engineer, discretization methods for Maxwell's equations, finite element methods, numerical linear algebra and data structures are all part of the modern toolkit for effective design and topology seems to have taken a back seat. On the other hand, to an engineer from a half-century ago, the connection between electromagnetic theory and topology would be considered "obvious" by considering Kirchhoff's laws and circuit theory in the light of Maxwell's electromagnetic theory. To this older electrical engineer, topology would be considered part of the engineer's art with little connection to computation beyond what Maxwell and Kirchhoff would have regarded as computation. A mathematician could snicker at the two engineers and proclaim that all is trivial once one gets to the bottom of algebraic topology. Indeed the present book can be regarded as a logical consequence for computational electromagnetism of Eilenberg and Steenrod's *Foundations of Algebraic Topology* [ES52], Whitney's *Geometric Integration Theory* [Whi57] and some differential topology. Of course, this would not daunt the older engineer who accomplished his task before mathematicians and philosophers came in to lay the foundations.

The three points of view described above expose connections between pairs of each of the three fields, so it is natural to ask why it is important to put all three together in one book. The answer is stated quite simply in the context of the three characters mentioned above. In a modern "design automation" environment, it is necessary to take the art of the old engineer, reduce it to a science as much as possible, and then turn that into a numerical computation. For the purposes of computation, we need to feed a geometric model of a device such as a motor or circuit board, along with material properties, to a program which exploits algebraic topology in order to extract a simple circuit model from a horrifically complicated description in terms of partial differential equations and boundary value problems. Cohomology and Hodge theory on manifolds with boundary are the bridge between Maxwell's equations and the lumped parameters of circuit theory, but engineers need software that can reliably make this connection in an accurate manner. This book exploits developments in algebraic topology since the time of Maxwell to provide a framework for linking data structures, algorithms, and computation to topological aspects of 3-dimensional electromagnetic boundary value problems. More simply, we develop the link between Maxwell and a modern topological approach to algorithms for the analysis of electromagnetic devices.

To see why this is a natural evolution, we should review some facts from recent history. First, there is Moore's law, which is not a physical law but the observation that computer processing power has been doubling every eighteen months. In practical terms this means that in the year 2003 the video game played by five-year-old playing had the same floating-point capability as the largest supercomputer 15 years earlier. Although the current use of the term "computer" did not exist in the English language before 1950, Moore's law can be extrapolated back in time to vacuum tube computers, relay computers, and mechanical computing machines of the 1920's. Moving forward in time, the economics of building computers will bring this exponential increase to a halt before physics predicts the demise of Moore's law, but we are confident this trend will continue for at least another decade. Hence we should consider scientific computing and computational electromagnetics in this light.

The second set of facts we need to review concern the evolution of the tools used to solve elliptic boundary value problems. This story starts with Dirichlet's principle, asserting the existence of a minimizer for a quadratic functional whose Euler–Lagrange equation is Laplace's equation. Riemann used it effectively in his theory of analytic functions, but Weierstrass later put it into disrepute with his counterexamples. Hilbert rescued it with the concept of a minimizing sequence, and in the process modern functional analysis took a great step forward. From the point of view of finite element analysis, the story really starts with Courant, who in the 1920's suggested triangulating the underlying domain, using piecewise-polynomial trial functions for Ritz's method and producing a minimizing sequence by subdividing the triangulation. Courant had a constructive proof in mind, but three decades later his idea was the basis of the finite element method. Issues of adaptive mesh refinement can be interpreted as an attempt to produce a best approximation for a fixed number of degrees of freedom, as the number of degrees of freedom increase. In the electrical engineering of the 1960's, the finite element method started making an impact in the area of two-dimensional static problems that could be formulated in terms of a scalar potential or stream function. With the development of computer graphics in the 1970's, electrical engineers were beginning to turn their attention to the representation of vector fields, three-dimensional problems, and the adaptive generation of finite element meshes. At the same time, it is somewhat unfortunate that the essence of electromagnetic theory seen in Faraday and Maxwell's admirable qualitative spatial reasoning was lost under the vast amounts of numerical data generated by computer. In the 1980's came the realization that differential form methods could be translated to the discrete setting and that the hard work had already been done by André Weil and Hassler Whitney in the 1950's, but this point of view was a little slow to catch on. Technology transfer from mathematics to engineering eventually happened, since all of this mathematics from the 1950's was set in terms of simplicial complexes which fit hand in glove with the data structures of finite element analysis.

Before outlining the book in detail, there is one more observation to make about the process of automating the topological aspects that were once considered to be the engineer's art. Not only has the exponential increase in computing power given us the means to tackle larger and higher dimensional problems, but it has fundamentally changed the way we interact with computers. It took less than twenty years from "submitting a job" with a stack of punched cards at the university computing center to simulating an electromagnetic field in a personal virtual reality environment. With the continuing evolution of three-dimensional, real-time video games, we are assured of improved environments for having computers deal with the topological aspects of electromagnetic design. The task at hand is to identify the interactions between electromagnetics and algebraic topology that can have the greatest impact on formalizing the design engineer's intuition so that computers can be integrated more effectively into the design process.

Outline of Book. Chapter 1 develops homology and cohomology in the context of vector calculus, while suppressing the formalism of exterior algebra and differential forms. This enables practicing engineers to appreciate the relevance of the material with minimal effort. Although Gauss, Helmholtz, Kirchhoff and Maxwell recognized that topology plays a pivotal role in the formulation of electromagnetic boundary value problems, it is still a largely unexploited tool in problem formulation and computational methods for electromagnetic fields. Most historians agree that Poincaré and Betti wrote the seminal papers on what is now known as algebraic topology. However, it is also clear that they stood on the shoulders of Riemann and Listing. A glimpse into the first chapter of [Max91] shows that these same giants were under the feet of Maxwell. Correspondence between Maxwell and Tait reveals that Maxwell consciously avoided both Grassmann's exterior algebra and Hamilton's quaternions as a formalism for electromagnetism in order to avoid ideological debates. Credit is usually given to Oliver Heaviside for fitting Maxwell's equations into a notation accessible to engineers. Hence it is fair to say that the wonderful insights into three-dimensional topology found in Maxwell's treatise have never been exploited effectively by engineers. Thus our first chapter is a tunnel from some of the heuristic topological instincts of engineers to the commutative algebraic structures that can be extracted from the data structures found in electromagnetic field analysis software. A mathematician would make this all rigorous by appealing to the formalism of differentiable manifolds and differential forms. We leave the reader the luxury of seeing how this happens in a Mathematical Appendix.

Chapter 2 underlines the notion of a quasistatic electromagnetic field in the context Maxwell's equations. Quasistatics is an engineer's ticket to elliptic boundary value problems, variational principles leading to numerical algorithms, and the finite element method. We make certain physical assumptions in order to formulate the quasistatic problem, and the reader gets to see how circuit theory in the sense of Kirchhoff arises in the context of quasistatic boundary value problems. Besides promoting the boundary value problem point of view, the variational principles discussed in chapter 2 tie duality theorems for manifolds with boundary to the lumped parameters of circuit theory.

Having had a intuitive glimpse into the uses of duality theorems for manifolds with boundary in the first two chapters, Chapter 3 goes on to formalize some of the underlying ideas. After presenting the traditional Poincaré and Lefschetz duality theorems in the context of electromagnetics and circuit theory, we move to Alexander duality and present it in the context of linking numbers and magnetic scalar potentials. This approach is closest to Gauss' understanding of the matter and is completely natural in the context of magnetoquasistatics. Finally, for subsets of three-dimensional Euclidean space that have a continuous retraction into their interiors, we show that the absolute and relative (modulo boundary) homology and cohomology groups, as commutative groups, are torsion-free. This is significant for two reasons. First, it tells us why coming up with simple examples of torsion phenomena in three dimensions is a bit tricky, and second, it paves the way to using integer arithmetic in algorithms which would otherwise be susceptible to rounding error if implemented with floating point operations. With this result we are ready to return to the primary concerns of the engineer.

In Chapter 4 we finally arrive at the finite element method. It is introduced in the context of Laplace's equation and a simplicial mesh. The simplicial techniques used in topology are shown to translate into effective numerical algorithms that are naturally phrased in terms of the data structures encountered in finite element analysis. Although this opens the door to many relatively recent developments in computational electromagnetics, we focus on how the structures of homology and cohomology arise in the context of finite element algorithms for computing 3-dimensional electric and magnetic fields. In this way, the effectiveness of algebraic topology can be appreciated in a well-studied computational setting. Along the way we also get to see how the Euler characteristic is an effective tool in the analysis of algorithms.

One of the main strengths of the book comes to center stage in Chapter 5. This chapter addresses the problem of coupling magnetic scalar potentials in multiply-connected regions to stream functions which describe currents confined to conducting surfaces. This problem is considered in detail and the topological aspects are followed from the problem formulation stage through to the matrix equations arising from the finite element discretization. In practice this problem arises in non-destructive evaluation of aircraft wings, pipes, and other places. This problem is unique in that it is a three-dimensional magnetoquasistatic problem which admits a formulation solely in terms of scalar potentials, yet the topological aspects can be formulated in full generality while the overall formulation is sufficiently simple that it can be presented concisely. This chapter builds on all of the concepts developed in previous chapters, and is an ideal playground for illustrating how the tools of homological algebra (long exact sequences, duality theorems, etc.) are essential from problem formulation to interpretation of the resulting matrix equations.

Chapter 5 is self-contained except that one fundamental issue is acknowledged but sidestepped up to this point in the book. That issue is computation of cuts for magnetic scalar potentials. This is a deep issue since the simplest general definition of a cut is a realization, as an embedded orientable manifold with boundary, of an element of the second homology group of a region modulo its boundary. Poincaré and Maxwell took the existence of cuts for granted, and it was Pontryagin and Thom who, in different levels of generality, pointed out the need for an existence proof and gave a general framework for realizing homology classes as manifolds in the case that there is such a realization. It is ironic that historically, this question was avoided until the tools for its resolution were developed. For our purposes, an existence proof is given in the Mathematical Appendix, and the actual algorithm for computing a set of cuts realizing a basis for the second homology group is given in chapter six.

Chapter 6 bridges the gap between the existence of cuts and their realization as piecewise-linear manifolds which are sub-complexes of a finite element mesh (considered as a simplicial complex). Any algorithm to perform this task is useful only if some stringent complexity requirements are met. Typically, on a given mesh, a magnetic scalar potential requires about an order of magnitude less work to compute than computing the magnetic field directly. Hence if the computation of cuts is not comparable to the computation of a static solution of a scalar potential subject to linear constitutive laws, the use of scalar potentials in multiply-connected regions is not feasible for time-varying and/or nonlinear problems. We present an algorithm that involves the formulation and finite element solution of a Poisson-like equation, and additional algorithms that involve only integer arithmetic. We then have a favorable expression of the overall complexity in terms of a familiar finite element solution and the reordering and solution of a large sparse integer matrix equation arising for homology computation. This fills in the difficult gap left over from Chapter 5.

Chapter 7, the final chapter, steps back and considers the techniques of homological algebra in the context of the variational principles used in the finite element analysis of quasistatic electromagnetic fields. The message of this chapter is that the formalism of homology, and cohomology theory via differential form methods, are essential for revealing the conceptual elegance of variational methods in electromagnetism as well as providing a framework for software development. In order to get this across, a paradigm variational problem is formulated which includes as special cases all of the variational principles considered in earlier chapters. All the topological aspects considered in earlier chapters are then seen in the light of the homology and cohomology groups arising in the analysis of this paradigm problem. Because the paradigm problem is n-dimensional, this chapter no longer emphasizes the more visual and intuitive aspects, but exploits the formalism of differential forms in order to make connections to Hodge theory on manifolds with boundary, and variational methods for quasilinear elliptic partial differential equations. The engineer's topological intuition has now been obscured, but we gain a paradigm variational problem for which topological aspects which lead to circuit models are reduced by Whitney form discretization to computations involving well-understood algorithms.

The Mathematical Appendix serves several purposes. First, it contains results that make the book more mathematically self-contained. These results make the algebraic aspects accessible to the uninitiated, tie differential forms to cohomology, make clear what aspects of cohomology theory depend on the metric or constitutive law, and which do not. Second, certain results, such as the proof of the existence of cuts, are presented. This existence proof points to an algorithm for finding cuts, but involves tools from algebraic topology not found in introductory treatments. Having this material in an appendix makes the chapters of the book more independent.

Having stated the purpose of the book and outlined its contents, it is useful to list several problems not treated in this book. They represent future work which may be fruitful:

(1) Whitney forms and Whitney form discretizations of helicity functionals, their functional determinants, and applications to impedance tomography. There is already a nice exposition on Whitney forms accessible to engineers [Bos98].
(2) Lower central series of the fundamental group and, in three dimensions, the equivalent data given by Massey products in the cohomology ring. This algebraic structure contains more information than homology groups but, unlike the fundamental group, the computation of the lower central series can be done in polynomial time and gives insight into computational complexity of certain sparse matrix techniques associated with homology calculations.
(3) Additional constraints on cuts. Although we present a robust algorithm for computing cuts for magnetic scalar potentials, one may consider whether, topologically speaking, these cuts are the simplest possible. Engineers should not have to care about this, but the problem is very interesting as it relates to the computation of the Thurston norm on homology. Furthermore, if one introduces force constraints into the magnetoquasistatic problems considered in this book, the problem is related to the physics of "force-free magnetic fields" and has applications from practical magnet design to understanding the solar corona.
(4) Common historical roots between electromagnetism, computation and topology. Electromagnetic theory developed alongside topology in the works of Gauss, Weber, Möbius and Riemann. These pioneers also had a great influence on each other which is not well documented. In addition, Courant's paper, which lead to the finite element method, was written when triangulations of manifolds were the order of the day, and about the time when simplicial techniques in topology were undergoing rapid development in Göttingen.

We hope that the connections made in this book will inspire the reader to take this material beyond the stated purpose of developing the connection between algebraic structures in topology and methods for 3-dimensional electric and magnetic field computation.

Any problem which is nonlinear in character… or whose structure is initially defined in the large, is likely to require considerations of topology and group theory in order to arrive at its meaning and its solution.

Marston Morse, The Calculus of Variations in the Large, 1934

1

From Vector Calculus to Algebraic Topology

1A. Chains, Cochains and Integration

Homology theory reduces topological problems that arise in the use of the classical integral theorems of vector analysis to more easily resolved algebraic problems. Stokes' theorem on manifolds, which may be considered the fundamental theorem of multivariable calculus, is the generalization of these classical integral theorems. To appreciate how these topological problems arise, the process of integration must be reinterpreted algebraically.

Given an n-dimensional region Ω, we will consider the set $C_p(\Omega)$ of all possible p-dimensional objects over which a p-fold integration can be performed. Here it is understood that $0 \leq p \leq n$ and that a 0-fold integration is the sum of values of a function evaluated on a finite set of points. The elements of $C_p(\Omega)$, called *p-chains*, start out conceptually as p-dimensional surfaces, but in order to serve their intended function they must be more than that, for in evaluating integrals it is essential to associate an orientation to a chain. Likewise the idea of an orientation is essential for defining the oriented boundary of a chain (Figure 1.1).

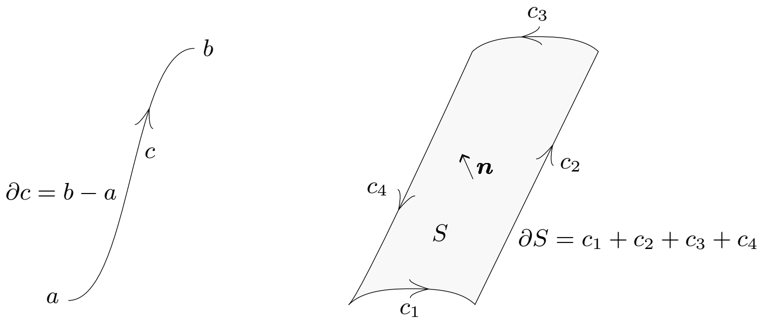

Figure 1.1. Left: a 1-chain. Right: a 2-chain.

At the very least, then, we wish to ensure that our set of chains $C_p(\omega)$ is closed under orientation reversal: for each $c \in C_p(\Omega)$ there is also $-c \in C_p(\Omega)$.

The set of integrands of p-fold integrals is called the set of *p-cochains* (or p-forms) and is denoted by $C^p(\Omega)$. For a chain $c \in C_p(\Omega)$ and a cochain $\omega \in C^p(\Omega)$, the integral of ω over c is denoted by $\int_c \omega$, and integration can be regarded as a mapping

$$\int : C_p(\Omega) \times C^p(\Omega) \to \mathbb{R}, \quad \text{for } \ 0 \le p \le n,$$

where $\mathbb{R}$ is the set of real numbers. Integration with respect to p-forms is a linear operation: given $a_1, a_2 \in \mathbb{R}$, $\omega_1, \omega_2 \in C^p(\Omega)$ and $c \in C_p(\Omega)$, we have

$$\int_c a_1\omega_1 + a_2\omega_2 = a_1 \int_c \omega_1 + a_2 \int_c \omega_2.$$

Thus $C^p(\Omega)$ may be regarded as a vector space, which we denote by $C^p(\Omega, \mathbb{R})$. Reversing the orientation of a chain means that integrals over that chain acquire the opposite sign:

$$\int_{-c} \omega = -\int_c \omega.$$

More generally, it is convenient to regard $C_p(\Omega)$ as having some algebraic structure — for example, an abelian group structure, as follows:

Example 1.1 Chains on a transformer. This example is inspired by electrical transformers, though understanding of a transformer is not essential for understanding the example. A current-carrying coil with n turns is wound around a toroidal piece of magnetic core material. The coil can be considered as a 1-chain, and it behaves in some ways as a multiple of another 1-chain c', a single loop going around the core once (see Figure 1.2). For instance, the voltage V_c

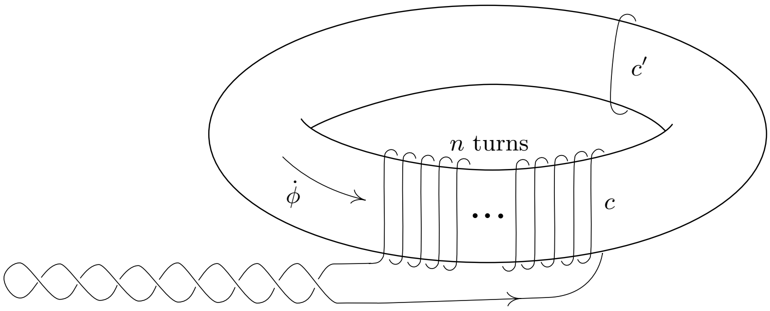

Figure 1.2. Windings on a solid toroidal transformer core. A 1-chain c in $C_1(\mathbb{R}^3 - \text{core})$ can be considered as a multiple of the 1-chain c'.

induced in loop c can be calculated in terms of the voltage of loop c' from the electric field $\boldsymbol{E}$ as

$$V_c = \int_c \boldsymbol{E} \cdot \boldsymbol{t}\, dl = \int_{nc'} \boldsymbol{E} \cdot \boldsymbol{t}\, dl = n \int_{c'} \boldsymbol{E} \cdot \boldsymbol{t}\, dl = nV_{c'},$$

where $\boldsymbol{t}$ is the unit vector tangential to c (or c'). □

For this reason it is convenient to regard as a 1-chain any integer multiple of a 1-chain, or even any linear combination of 1-chains. That is, we insist that our set of 1-chains be closed under chain addition (we had already made it closed under inversion or reversal). Moreover we insist that the properties of an abelian group (written additively) should be satisfied: for 1-chains c, c', c'', we have

$$c + (-c) = 0, \quad c + 0 = c, \quad c + c' = c' + c, \quad c + (c' + c'') = (c + c') + c''.$$

Given any n-dimensional region Ω, the set of "naive" p-chains $C_p(\Omega)$ can be extended to an abelian group by this process, the result being the set of all linear combinations of elements of $C_p(\Omega)$ with coefficients in $\mathbb{Z}$ (the integers). This group is denoted by $C_p(\Omega, \mathbb{Z})$ and called the *group of p-chains with coefficients in* $\mathbb{Z}$.

If linear combinations of p-chains with coefficients in the field $\mathbb{R}$ are used in the construction above, the set of p-chains can be regarded as a vector space. This vector space, denoted by $C_p(\Omega, \mathbb{R})$ and called the *p-chains with coefficients in* $\mathbb{R}$, will be used extensively. In this case, for $a_1, a_2 \in \mathbb{R}$, $c_1, c_2 \in C_p(\Omega, \mathbb{R})$, $\omega \in C^p(\Omega, \mathbb{R})$,

$$\int_{a_1 c_1 + a_2 c_2} \omega = a_1 \int_{c_1} \omega + a_2 \int_{c_2} \omega.$$

In a similar fashion, taking a ring R and forming linear combinations of p-chains with coefficients in R, we have an R-module $C_p(\Omega, R)$, called the *p-chains with coefficients in R*. This construction has the previous two as special cases. It is possible to construct analogous groups for p-cochains, but we need not do so at the moment. Knowledge of rings and modules is not crucial at this point; rather the construction of $C_p(\Omega, R)$ is intended to illustrate how the notation is developed.

For coefficients in $\mathbb{R}$, the operation of integration can be regarded as a bilinear pairing between p-chains and p-forms. Furthermore, for reasonable p-chains and p-forms this bilinear pairing for integration is nondegenerate. That is,

$$\text{if } \int_c \omega = 0 \text{ for all } c \in C_p(\Omega), \text{ then } \omega = 0$$

and

$$\text{if } \int_c \omega = 0 \text{ for all } \omega \in C^p(\Omega), \text{ then } c = 0.$$

Although this statement requires a sophisticated discretization procedure and limiting argument for its justification [Whi57, dR73], it is plausible and simple to understand.

In conclusion, it is important to regard $C_p(\Omega)$ and $C^p(\Omega)$ as vector spaces and to consider integration as a bilinear pairing between them. In order to reinforce this point of view, the process of integration will be written using the linear space notation

$$\int_c \omega = [c, \omega];$$

that is, $C^p(\Omega)$ is to be considered the dual space of $C_p(\Omega)$.

1B. Integral Laws and Homology

Consider the fundamental theorem of calculus,

$$\int_c \frac{\partial f}{\partial x}\,dx = f(b) - f(a), \quad \text{where } c = [a,b] \in C_1(\mathbb{R}^1).$$

Its analogs for two-dimensional surfaces Ω are:

$$\int_c \operatorname{grad}\phi \cdot \boldsymbol{t}\,dl = \phi(p_2) - \phi(p_1) \quad \text{and} \quad \int_S \operatorname{curl}\boldsymbol{F}\cdot\boldsymbol{n}\,dS = \int_{\partial S}\boldsymbol{F}\cdot\boldsymbol{t}\,dl,$$

where $c \in C_1(\Omega)$, $\partial c = p_2 - p_1$, and $S \in C_2(\Omega)$. In three-dimensional vector analysis ($\Omega \subset \mathbb{R}^3$) we have

$$\int_c \operatorname{grad}\phi \cdot \boldsymbol{t}\,dl = \phi(p_2) - \phi(p_1), \qquad \int_S \operatorname{curl}\boldsymbol{F}\cdot\boldsymbol{n}\,dS = \int_{\partial S}\boldsymbol{F}\cdot\boldsymbol{t}\,dl,$$

$$\int_V \operatorname{div}\boldsymbol{F}\,dV = \int_{\partial V}\boldsymbol{F}\cdot\boldsymbol{n}\,dS,$$

where $c \in C_1(\Omega)$, $\partial c = p_2 - p_1$, $S \in C_2(\Omega)$, and $V \in C_3(\Omega)$. Note that here we are regarding p-chains as point sets but retaining information about their orientation.

These integral theorems, along with four-dimensional versions that arise in covariant formulations of electromagnetics, are special instances of the general result called Stokes' theorem on manifolds. This result, discussed at length in Section MA-H (page 232), takes the form

$$\int_c d\omega = \int_{\partial c}\omega,$$

where the linear operators for boundary (∂) and exterior derivative (d) are defined in terms of direct sums:

$$\partial : \bigoplus_p C_p(\Omega) \to \bigoplus_p C_{p-1}(\Omega), \qquad d : \bigoplus_p C^{p-1}(\Omega) \to \bigoplus_p C^p(\Omega).$$

When p-forms are called p-cochains, d is called the coboundary operator. For an n-dimensional region Ω the following definition is made:

$$C^p(\Omega) = 0 \;\text{ for } p < 0, \qquad C_p(\Omega) = 0 \text{ for } p > n.$$

In this way, the boundary operator on p-chains has an intuitive meaning which carries over from vector analysis. On the other hand, the exterior derivative must be regarded as the operator which makes Stokes' theorem true. When a formal definition of the exterior derivative is given in a later chapter, it will be a simple computation to verify the special cases listed above.

For the time being, let the restriction of the boundary operator to p-chains be denoted by ∂_p and the restriction of the exterior derivative to p-forms be denoted by d^p. Thus

$$\partial_p : C_p(\Omega) \to C_{p-1}(\Omega) \quad \text{and} \quad d^p : C^p(\Omega) \to C^{p+1}(\Omega).$$

Considering various n-dimensional regions Ω and p-chains for various values of p, it is apparent that the boundary of a boundary is zero

$$(\partial_p \partial_{p+1})c = 0 \;\text{ for all } c \in C_{p+1}(\Omega).$$