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Finite Packing and Covering

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Congruent Domains in the Euclidean Plane

Let K be a convex domain. According to the classical result of L. Fejes Tóth [FTL1950], the density of a packing of congruent copies of K in a hexagon cannot be denser than the density of K inside the circumscribed hexagon with minimal area. Besides this statement, we verify that the same density estimate holds for any convex container provided the number of copies is high enough. In addition, we show that if K is a centrally symmetric domain then the inradius and circumradius of the optimal convex container cannot be too different. Following L. Fejes Tóth [FTL1950] in case of coverings, the analogous density estimate is verified under the "noncrossing" assumption, which essentially says that the boundaries of any two congruent copies intersect in two points. In case of both packings and coverings, congruent copies can be replaced by similar copies of not too different sizes. Finally, we verify the hexagon bound for coverings by congruent fat ellipses even without the noncrossing assumption, a result due to A. Heppes.

Concerning the perimeter, we show that the convex domain of minimal perimeter containing n nonoverlapping congruent copies of K gets arbitrarily close to being a circular disc for large n. However, if the perimeter of the compact convex set D covered by n congruent copies of K is maximal then D is close to being a segment for large n.

1.1. Periodic and Finite Arrangements

Let K be a convex domain. Given an arrangement of congruent copies of K that is periodic with respect to some lattice Λ (see Section A.13) and given m equivalence classes, it is natural to call $m \cdot A(K)/\det \Lambda$ the density of the arrangement. We define the *packing density* $\delta(K)$ to be the supremum of the densities of periodic packings of congruent copies of K and the *covering density* to be the infimum of the densities of periodic coverings by congruent copies of K. In addition, we define $\Delta(K) = A(K)/\delta(K)$ and $\Theta(K) = A(K)/\vartheta(K)$.

It is not hard to show that optimal clusters asymptotically provide the same densities as periodic arrangements (see Lemma 1.1.2). Our main result is that, in the planar case, finite packings are not denser (asymptotically) than periodic packings, and the analogous statement holds for coverings. We note that this is a planar phenomenon: Say, if $d \geq 3$ and K is a right cylinder whose base is a $(d-1)$-ball, then linear arrangements are of density one, whereas any periodic packing is of density at most δ' for some $\delta' < 1$, and any periodic covering is of density at least ϑ' for some $\vartheta' > 1$ (see Lemma 7.2.5).

Theorem 1.1.1. *Let K be a convex domain, and let n tend to infinity.*

(i) If D_n is a convex domain of minimal area containing n nonoverlapping congruent copies of K then $A(D_n) \sim n \cdot \Delta(K)$.
(ii) If $\widetilde{D}_n$ is a convex domain of maximal area that can be covered by n congruent copies of K then $A(\widetilde{D}_n) \sim n \cdot \Theta(K)$.

Since periodic arrangements correspond canonically to finite arrangements on tori (see Section A.13), $\Delta(K)$ is the infimum of $V(T)/m$ over all tori T and integers m such that there exists a packing of m embedded copies of K on T, and $\Theta(K)$ is the supremum of $V(T)/m$ over all tori T and integers m such that there exists a covering of T by m embedded copies of K. The first step towards verifying Theorem 1.1.1 is the case of clusters.

Lemma 1.1.2. *Given convex domains K and D with $r(D) > R(K)$, let N be the maximal number of nonoverlapping congruent copies of K inside D, and let M be minimal number of congruent copies of K that cover D. Then*

(i) $\left(1 + \frac{R(K)}{r(D)}\right)^2 \cdot A(D) \geq N \cdot \Delta(K) \geq \left(1 - \frac{R(K)}{r(D)}\right)^2 \cdot A(D)$;
(ii) $\left(1 + \frac{R(K)}{r(D)}\right)^2 \cdot A(D) \geq M \cdot \Theta(K) \geq \left(1 - \frac{R(K)}{r(D)}\right)^2 \cdot A(D)$.

Remark. Instead of the upper bound in (i), we actually prove the stronger estimate $A(D + R(K)\,B^2) \geq N \cdot \Delta(K)$.

Proof. We place K and D in a way that $K \subset R(K)B^2 \subset D$. In particular, assuming that K' is congruent to K, if the circumcentre c of K' lies outside $D + R(K)B^2$ then K' avoids D, and if $c \in (1 - R(K)/r(D))D$ then $K' \subset D$. Given a torus T, we write the same symbol to denote a convex domain in $\mathbb{R}^2$ and its embedded image on T.

We present the proof only for packings because the case of coverings is completely analogous. Let $T = \mathbb{R}^2/\Lambda$ be any torus satisfying that

$C = D + R(K)\,B^2$ embeds isometrically into T, and let $K_1, \ldots, K_m$ be the maximal number of nonoverlapping embedded copies of K on T. Writing x_i to denote the circumcentre of K_i, we have (see (A.50))

$$\int_T \#((C + x) \cap \{x_1, \ldots, x_m\})\, dx = m \cdot A(C). \tag{1.1}$$

Thus there exists a translate $C + x$ that contains at most $k \leq m \cdot A(C)/A(T)$ points out of $x_1, \ldots, x_m$, say, the points $x_{i_1}, \ldots, x_{i_k}$. After replacing $K_{i_1}, \ldots, K_{i_k}$ by the N nonoverlapping embedded copies of K contained in $x + D$, we obtain a packing of $m - k + N$ embedded copies of K on T. In particular, $N \leq k$ follows by the maximality of m. We conclude

$$A(D + R(K)\,B^2) \geq N \cdot \Delta(K),$$

which in turn yields the upper bound in (i).

Turning to (ii), we let $\lambda < 1$ satisfy $\lambda \cdot A(C)/\Delta(K) > \lceil A(C)/\Delta(K) \rceil - 1$. It follows by the definition of $\Delta(K)$ that there exist a torus $T = \mathbb{R}^2/\Lambda$ and m nonoverlapping embedded copies $K_1, \ldots, K_m$ of K on T satisfying $A(T) < \lambda^{-1} m \Delta(K)$, and D embeds isometrically into T. We define $C = (1 - R(K)/r(D))D$; hence, (1.1) yields that some translate $C + x$ contains at least $m \cdot A(C)/A(T)$ points out of the circumcentres of $K_1, \ldots, K_m$. We may assume that these points are the circumcentres of $K_1, \ldots, K_l$; therefore, $l \geq \lambda \cdot A(C)/\Delta(K)$ and $K_1, \ldots, K_l$ are contained in $D + x$. Thus $N \geq l \geq A(C)/\Delta(K)$ by the definition of λ, completing the proof of Lemma 1.1.2. □

Proof of Theorem 1.1.1. We present the argument only for packings because just the obvious changes are needed for the case of coverings. In the following the implied constant in $O(\cdot)$ always depends only on K.

Theorem 1.1.1 for packings follows from the following statement: If $\varepsilon > 0$ is small, and $n > 1/\varepsilon^5$ then

$$A(D_n) = (1 + O(\varepsilon)) \cdot n\Delta(K). \tag{1.2}$$

Dense clusters show (see Lemma 1.1.2) that

$$A(D_n) \leq (1 + O(\varepsilon)) \cdot n\Delta(K).$$

Therefore, it is sufficient to verify that

$$A(D_n) \geq (1 - O(\varepsilon)) \cdot n\Delta(K). \tag{1.3}$$

If $r(D_n) > 1/\varepsilon$ then (1.3) follows from Lemma 1.1.2. Thus we assume that $r(D_n) \leq 1/\varepsilon$, a case that requires a more involved argument. We actually prove that there exists a rectangle R that contains certain N congruent copies of K, where

$$\frac{A(R)}{N} \leq (1 + O(\varepsilon)) \cdot \frac{A(D_n)}{n}. \tag{1.4}$$

Since the minimal width w of D_n is at most $3/\varepsilon$ according to the Steinhagen inequality (Theorem A.8.2), there exists a rectangle $\widetilde{R}$ such that its sides touch D_n, and two parallel sides of $\widetilde{R}$ are of length w. We say that these sides are vertical; hence, D_n has a vertical section of length w. Writing l to denote the length of the horizontal sides, we have $A(D_n) \geq wl/2$. For $k = \lceil 1/\varepsilon \rceil$, we decompose $\widetilde{R}$ into k^3 congruent rectangles $R_1, \ldots, R_{k^3}$ in this order, where the vertical sides of R_i are of length w and the horizontal sides are of length l/k^3.

Out of the circumcentres of the n nonoverlapping congruent copies of K that lie in D_n, let n_i be contained in R_i. Now the total area of $R_1, \ldots, R_{k^2+1}$ and of $R_{k^3-k^2}, \ldots, R_{k^3}$ is

$$\left(1 + \frac{1}{k^2}\right) \frac{2wl}{k} \leq (1 + O(\varepsilon))\, 4\varepsilon A(D_n) \leq (1 + O(\varepsilon))\, 4\Delta(K) \cdot \varepsilon n,$$

and hence $\sum_{i=k^2+1}^{k^3-k^2} n_i \geq (1 - O(\varepsilon))\, n$. In particular, there exists some index j such that $k^2 + 1 \leq j \leq k^3 - k^2$ and

$$\frac{A(R_j \cap D_n)}{n_j} \leq (1 + O(\varepsilon)) \cdot \frac{A(D_n)}{n}. \tag{1.5}$$

Let R' be the rectangle whose sides are vertical and horizontal, with each touching $R_j \cap D_n$. We write a to denote the common length of the vertical sides of R', which readily satisfies $a \geq 2r(K)$. Since $w/k^2 < 4\varepsilon$, we deduce that $R_j \cap D_n$ contains a rectangle whose horizontal side is of length l/k^3, and the vertical side is of length $a - 8\varepsilon$. In particular, $A(R')$ is at most $(1 + O(\varepsilon))A(R_j \cap D_n)$. Finally, the rectangle R whose horizontal sides are of length $l/k^3 + 2R(K)$ and vertical sides are of length a contains $N = n_j$ nonoverlapping congruent copies of K. Now $3l/\varepsilon \geq n\, A(K)$ yields $l/k^3 \geq 1/[4A(K)\varepsilon]$. Thus we conclude (1.4) by (1.5).

Since the arrangement in R induces a periodic packing of K, (1.4) readily yields (1.3) and hence Theorem 1.1.1 as well. $\square$

Remark 1.1.3. Given a strictly convex domain K, if D_n is a convex domain with minimal area that contains n nonoverlapping congruent copies of K then $r(D_n)$ tends to infinity.

We sketch the argument for Remark 1.1.3: We suppose indirectly that there exists a subsequence of $\{r(D_n)\}$ that is bounded by some $\omega > 0$. For any $\varepsilon > 0$, the proof of Theorem 1.1.1 yields a parallel strip Σ_ε and a packing

Ξ of congruent copies of K inside Σ_ε such that the packing is periodic with respect to a vector parallel to Σ_ε, the width of Σ_ε is at most 3ω, and the density of the packing Ξ inside Σ_ε is at least $(1-\varepsilon)\,\delta(K)$. We reflect this arrangement through one of the lines bounding Σ_ε and write Ξ' to denote the image of Ξ. Because K is strictly convex, there exist positive ν_1 and ν_2 depending only on K with the following property: Translating the packing Ξ' first parallel to Σ_ε by a vector of length ν_1, then towards Σ_ε orthogonally by a vector of length ν_2, we obtain an arrangement Ξ'' such that the union of Ξ and Ξ'' forms a packing. If $\varepsilon < \nu_2/(2\omega)$ then the union of Ξ and Ξ'' determines a periodic packing in the plane whose density is larger than $\delta(K)$. This contradiction verifies that $r(D_n)$ tends to infinity.

Open Problems.

(i) Let K be a convex domain that is not a parallelogram. We write D_n ($\widetilde{D}_n$) to denote a convex domain with minimal (maximal) area that contains n nonoverlapping congruent copies of K (i.e., is covered by n congruent copies of K). Is

$$r(D_n),\, r(\widetilde{D}_n) > c\sqrt{n}$$

for a suitable positive constant c depending on K? If the answer is yes then the ratio $R(D_n)/r(D_n)$ stays bounded as n tends to infinity, and a similar property holds for $\widetilde{D}_n$.

For packings, various partial results support an affirmative answer: The statement holds if K is centrally symmetric (see Corollary 1.4.3) or the packing is translative (see Theorem 2.4.1). Strengthening the method of Remark 1.1.3 yields that $r(D_n) > c\sqrt[3]{n}$ holds if K is any strictly convex domain. For coverings, the statement holds if K is a fat ellipse (see Theorem 1.7.1) or if K is centrally symmetric and only translative coverings are allowed (see Corollary 2.8.2).

(ii) Is $\vartheta(K) \leq 2\pi/\sqrt{27} = 1.2091\ldots$ for any convex domain K; namely, is the covering density maximal for circular discs (see Theorem 1.7.1)?

D. Ismailescu [Ism1998] proved $\vartheta(K) \leq 1.2281\ldots$ for any convex domain K. However, $\vartheta(K) \leq 2\pi/\sqrt{27}$ if K is centrally symmetric (see L. Fejes Tóth [FTL1972]).

(iii) Does it hold for any convex domain that there exist a periodic packing whose density is the packing density and a periodic covering whose density is the covering density? It is known that there exist no optimal lattice arrangement for the typical convex domain (see G. Fejes Tóth and T. Zamfirescu [FTZ1994] and G. Fejes Tóth [FTG1995a]).

Comments. The packing and covering densities were originally introduced in the framework of infinite packing and covering of the space (see G. Fejes Tóth and W. Kuperberg [FTK1993]). Readily, $\Theta(K) \leq A(K) \leq \Delta(K)$. W.M. Schmidt [Sch1961] proved that $\Theta(K) = A(K)$ or $\Delta(K) = A(K)$ if and only if some congruent copies of K tile the plane (see also Lemma 7.2.5).

According to the hexagon bound of L. Fejes Tóth [FTL1950] (see Theorem 1.3.1), $\Delta(K)$ is at least the minimal area of circumscribed hexagons for any convex domain K, where equality holds if K is centrally symmetric. Concerning absolute lower bounds on the packing density, G. Kuperberg and W. Kuperberg [KuK1990] verified that $\delta(K) > \sqrt{3}/2 = 0.8660\ldots$ holds for any convex domain K. In addition a beautiful little theorem of W. Kuperberg [Kup1987] states that $\delta(K)/\vartheta(K) \geq 3/4$, where equality holds for circular discs. It is probably surprising but the packing density, $\pi/\sqrt{12} = 0.9068\ldots$ of the unit disc is not minimal among centrally symmetric convex domains, which is shown say by the regular octagon. By rounding off the corners of the regular octagon, K. Reinhardt [Rei1934] and K. Mahler [Mah1947] proposed a possible minimal shape whose density is $0.9024\ldots$. P. Tammela [Tam1970] proved that $\delta(K) > 0.8926$ for any centrally symmetric convex domain K.

Concerning coverings, D. Ismailescu [Ism1998] proved $\vartheta(K) \leq 1.2281\ldots$ for any convex domain K. For very long the only convex domains with known covering densities were the tiles (when the covering density is one), and circular discs (when the covering density is $2\pi/\sqrt{27}$ according to R. Kershner [Ker1939]; see also Corollary 5.1.2). Recently A. Heppes [Hep2003] showed that the covering density of any "fat ellipse" (when the ratio of the smaller axis to the greater axis is at least 0.86) is $2\pi/\sqrt{27}$ (see also Theorem 1.7.1). A substantial improvement is due to G. Fejes Tóth [FTG?b]: On the one hand [FTG?b] generalized A. Heppes' theorem to ellipses when the ratio of the smaller axis to the greater axis is at least 0.741. On the other hand if K is a centrally symmetric convex domain and $r(K)/R(K) \geq 0.933$ then [FTG?b] proves that $\Theta(K)$ is the maximal area of polygons with at most six sides inscribed into K. Readily if K is either type of the convex domains considered in [FTG?b], and $C \subset K$ is a convex domain that contains a centrally symmetric hexagon of area $\Theta(K)$ then $\Theta(C) = \Theta(K)$.

1.2. The Hexagon Bound for Packings Inside an Octagon

Given a convex domain K, we write $H(K)$ to denote a circumscribed convex polygon with at most six sides of minimal area. The aim of this section is to verify the *hexagon bound* for packings of congruent copies of K

inside a hexagon; namely, the density is at most $A(K)/A(H(K))$. Later we will prove the hexagon bound with respect to any convex container (see Theorem 1.4.1).

Theorem 1.2.1. *If a polygon D of at most eight sides contains $n \geq 2$ congruent copies of a given convex domain K then*

$$A(D) \geq n \cdot A(H(K)).$$

The main idea of the proof Theorem 1.2.1 is to define a cell decomposition of D into convex cells in a way such that each cell contains exactly one of the congruent copies of K; hence, the average number of sides of the cells is at most six according to the Euler formula. Then we verify that the minimal areas of circumscribed k-gons are convex functions of k (see Corollary 1.2.4), and we deduce that the average area of a cell is at least $A(H(K))$. Unfortunately, we cannot proceed exactly like this because no suitable cell decomposition of D may exist. In spite of this we can still save the essential properties of a cell decomposition (see Lemma 1.2.2) and verify the hexagon bound. Lemma 1.2.2 is presented in a rather general setting because of later applications.

Lemma 1.2.2. *Let D be a convex domain that contains the nonoverlapping convex domains $K_1, \ldots, K_n$, $n \geq 2$. Then there exist nonoverlapping convex domains $\Pi_1, \ldots, \Pi_n \subset D$ satisfying the following properties:*

(i) $K_i \subset \Pi_i$.
(ii) $\Pi_1, \ldots, \Pi_n$ *cover* ∂D.
(iii) Π_i *is bounded by* $k_i \geq 2$ *convex arcs that we call edges. The edges intersecting* $\operatorname{int} D$ *are segments, and the rest of the edges are the maximal convex arcs of* $\partial D \cap \Pi_i$.
(iv) The number b of edges contained in ∂D satisfy

$$\sum_{i=1}^{n}(6 - k_i) \geq b + 6.$$

In addition, if D is a polygon of at most eight sides and k_i^* denotes the number of sides of Π_i then $\sum_{i=1}^{n}(6 - k_i^*) \geq 0$.

Proof. Let $\Pi_1, \ldots, \Pi_n$ be nonoverlapping convex domains such that $K_i \subset \Pi_i \subset D$ and the total area covered by the convex domains $\Pi_1, \ldots, \Pi_n$ is maximal under these conditions. Since two nonoverlapping convex sets can be separated by a line, each Π_i is the intersection of a polygon P_i and D.

Now int $P_i \cap \partial D$ consists of finitely many convex arcs whose closures we call edges of Π_i. The rest of the edges of Π_i are the segments of the form $s \cap D$, where s is a side of P_i that intersects int D, and the vertices of Π_i are the endpoints of the edges.

It may happen that $\Pi_1, \dots, \Pi_n$ do not cover D, and we call the closure of a connected component of int $D \backslash \cup_{i=1}^n \Pi_i$ a hole. Let Q be a hole. Then there exists an edge e_1 of some Π_{i_1} such that e_1 intersects int D, and $e_1 \cap \partial Q$ contains a segment s_1, where we assume that s_1 is a maximal segment in $e_1 \cap \partial Q$. Since Π_{i_1} cannot be extended because of the maximality of $\sum A(\Pi_j)$, one endpoint v_2 of s_1 is contained in the relative interior of e_1; hence, $v_2 \in \mathrm{int} D$. Therefore, v_2 is the endpoint of an edge e_2 of some Π_{i_2} such that $e_2 \cap \partial Q$ contains a maximal segment s_2. Continuing this way we obtain that ∂Q is the union of segments $s_1, \dots, s_k$ with the following properties (where $s_0 = s_k$): s_j is contained in an edge e_j of some Π_{i_j}, and $s_j \cap s_{j-1}$ is a common endpoint $v_j \in \mathrm{int}\, D$ that is an endpoint of e_j and not of e_{j-1} for any $j = 1, \dots, k$; moreover, different s_i and s_j do not intersect otherwise. We deduce that Q is a convex polygon and $Q \subset \mathrm{int}\, D$.

Now (ii) readily follows; namely, $\Pi_1, \dots, \Pi_n$ cover ∂D. Next, we construct a related cell decomposition Σ of D by cells $\widetilde{\Pi}_1, \dots, \widetilde{\Pi}_n$. If there exists no hole then $\widetilde{\Pi}_i = \Pi_i$. Otherwise, let $\{Q_1, \dots, Q_m\}$ be the set of holes, let $q_j \in \mathrm{int}\, Q_j$, and we define $\widetilde{\Pi}_i$ to be the union of Π_i and all triangles of the form conv$\{q_j, s\}$ such that s is a side of Q_j and $s \subset \Pi_i$. In particular, the number of edges of Σ contained in $\widetilde{\Pi}_i$ is at least k_i; hence $\sum(6 - k_i) \geq b + 6$ is a consequence of Lemma A.5.9. If, in addition, D is a polygon of at most eight sides then $\sum k_i^* \leq 8 + \sum k_i$; thus $b \geq 2$ completes the proof of Lemma 1.2.2. □

Given the convex domain K, let $t_K(m)$ denote the minimal area of a circumscribed polygon of at most m sides for any $m \geq 3$. Next, we show that $t_K(m)$ is a convex function of m, more precisely, that $t_K(m)$ is even strictly convex if K is strictly convex.

Lemma 1.2.3. *If K is a strictly convex domain and $m \geq 4$ then*

$$t_K(m-1) + t_K(m+1) > 2\, t_K(m).$$

Proof. For any $m \geq 3$, we choose a circumscribed polygon Π_m of minimal area among the circumscribed polygons of at most m sides. Since K is strictly convex, Π_m is actually an m-gon, and each side of Π_m touches K at the midpoint of the side.

Let $m \geq 4$, and let $3 \leq k \leq l$ satisfy that $A(\Pi_k) + A(\Pi_l)$ is minimal under the condition $k + l = 2m$. We suppose that $k < m$ and seek a contradiction. The idea is to decrease the total area of Π_k and Π_l by interchanging certain sides. We write $p_1, \ldots, p_k$ and $q_1, \ldots, q_l$ to denote the midpoints of the sides of Π_k and Π_l, respectively, according to the clockwise orientation, and we write e_i and f_j to denote the side of Π_k and Π_l containing p_i and q_j, respectively. For $p, q \in \partial K$, let $[p, q)$ denote the semi open arc of ∂K, which starts at p and terminates at q according to the clockwise orientation, and the arc contains p and does not contain q. The k semi open convex arcs $[p_{i-1}, p_i)$ on ∂K (with $p_0 = p_k$) contain the $l \geq k + 2$ midpoints for Π_l, and hence either there exists $[p_{i-1}, p_i)$, which contains say q_1, q_2, q_3, or there exist two semi open arcs of the form $[p_{t-1}, p_t)$ such that each contains two midpoints from Π_l. In the first case, let Π'_{k+1} be obtained from Π_k by cutting off the vertex $e_{i-1} \cap e_i$ by aff f_2, and let Π''_{l-1} be obtained from Π_l by removing the side f_2, and hence aff $f_1 \cap$ aff f_3 is the new vertex of Π''_{l-1}. Then Π'_{k+1} and Π''_{l-1} have $k + 1$ and $l - 1$ sides, respectively, and $\Pi''_{l-1} \backslash \Pi_l$ is strictly contained in $\Pi'_{k+1} \backslash \Pi_k$. Therefore,

$$A(\Pi'_{k+1}) + A(\Pi''_{l-1}) < A(\Pi_k) + A(\Pi_l).$$

This is absurd, and hence we may assume that $q_1, q_2 \in [p_1, p_2)$ and $q_{j-1}, q_j \in [p_{i-1}, p_i)$ for $i \neq 2$. In this case let $\Pi'_{k'}$ be the circumscribed k'-gon defined by affine hulls of

$$e_1, f_2, \ldots, f_{j-1}, e_i, \ldots, e_k.$$

In addition, let $\Pi''_{l'}$ be the circumscribed l'-gon defined by affine hulls of

$$f_1, e_2, \ldots, e_{i-1}, f_j, \ldots, f_l;$$

thus $k' + l' = 2m$. When constructing $\Pi'_{k'}$ and $\Pi''_{l'}$, we remove the part of Π_k at the corner enclosed by e_1 and e_2 and cut off by $\partial \Pi_l$, and we add two nonoverlapping domains contained in this part (where one of the domains degenerates if $q_1 = p_1$). Because the situation is analogous at the corner of Π_k enclosed by e_{i-1} and e_i, we deduce that

$$A(\Pi'_{k'}) + A(\Pi''_{l'}) \leq A(\Pi_k) + A(\Pi_l).$$

The polygons were constructed in a way that $f_{j-2} \cap f_{j-1}$ is a common vertex for $\Pi'_{k'}$ and Π_l, whereas $f_j \cap f_{j-1}$ is a vertex for Π_l but not for $\Pi'_{k'}$, and hence q_{j-1} is not the midpoint of the side of $\Pi'_{k'}$ containing it. Therefore, there exists a circumscribed k'-gon whose area is less than $A(\Pi'_{k'})$, which contradicts the minimality of $A(\Pi_k) + A(\Pi_l)$. □

Given any convex domain K, we define $t_K(2) = (3/2)\, t_K(3)$. It is not hard to see that $t_K(3) \leq 2\, t_K(4)$; thus Lemma 1.2.3 and approximation yield that $t_K(m-1) + t_K(m+1) \geq 2\, t_K(m)$ holds for any $m \geq 3$. Defining $t_K(s)$ to be linear for $s \in [m-1, m]$, we deduce the following:

Corollary 1.2.4 (Dowker theorem for circumscribed polygons). *Given a convex domain K, the function $t_K(s)$ can be extended to be a convex and decreasing function for any real $s \geq 2$.*

After all these preparations, the proof of the main result is rather simple.

Proof of Theorem 1.2.1. Given the packing of n congruent copies of K inside the convex polygon D of at most eight sides, we construct the system of cells according to Lemma 1.2.2. It follows by Lemma 1.2.2 (iv) that the average number of sides is at most six; hence, the Dowker theorem (Corollary 1.2.4) yields that the average area of the cells is at least $A(H(K))$. In turn, we conclude Theorem 1.2.1. □

Comments. Theorem 1.2.1 is proved by L. Fejes Tóth [FTL1950], and Corollary 1.2.4 is due to C. H. Dowker [Dow1944].

1.3. The Hexagon Bound for Periodic Packings

In this section we estimate the packing density of a convex domain K. We recall that $H(K)$ denotes a circumscribed convex polygon with at most six sides of minimal area.

Theorem 1.3.1. *If K is a convex domain then $\Delta(K) \geq A(H(K))$. If in addition K is centrally symmetric then $\Delta(K) = A(H(K))$, and there exists a lattice packing of K that is a densest periodic packing.*

In light of the hexagon bound of Theorem 1.2.1, we only need to construct suitably efficient lattice packings of centrally symmetric domains. The first step is

Lemma 1.3.2. *If K is an o-symmetric convex domain then $H(K)$ can be chosen to be o-symmetric.*

Remark. There may exist circumscribed hexagons of minimal area that are not centrally symmetric. Say the regular octagon can be obtained from a square by cutting off four corners, and one circumscribed hexagon of minimal area is obtained by cutting off two neighbouring corners of the square.

Proof. We may assume that K is strictly convex, in which case we verify that any circumscribed hexagon H of minimal area is o-symmetric. The minimality of $A(H)$ yields that H has six sides that touch K in their midpoints. We write $p_1, \ldots, p_6$ to denote these midpoints in clockwise order and write l_i to denote the affine hull of the side touching at p_i. In addition, let p_i' and l_i' denote the reflected image of p_i and l_i, respectively, through o.

We suppose that p_1', is not among $p_2, \ldots, p_6$ for some i, and we seek a contradiction. We may assume that the shorter arc p_1p_2 of ∂K contains p_{j-1}' and p_j' for some $4 \leq j \leq 6$ and $p_j' \neq p_2$; hence, the shorter arc $p_1'p_2'$ contains p_{j-1} and p_j, and $p_j \neq p_2'$. Now let P be the circumscribed $(2j-4)$-gon determined by

$$l_2, \ldots, l_{j-1}, l_2', \ldots, l_{j-1}',$$

and let Q be the circumscribed $(16-2j)$-gon determined by

$$l_j, \ldots, l_6, l_1, l_j', \ldots, l_6', l_1'.$$

Thus $A(P) + A(Q) \leq 2A(H)$. Since p_1 is the midpoint of the side $l_1 \cap H$, it is not the midpoint of the side $l_1 \cap Q$; hence, Q is not a circumscribed $(16-2j)$-gon of minimal area. Therefore, the total area of P and of some circumscribed $(16-2j)$-gon is less than $2A(H)$, contradicting the Dowker theorem (Corollary 1.2.4). In turn, we conclude Lemma 1.3.2. □

Proof of Theorem 1.3.1. Let K be a convex domain. Applying Theorem 1.2.1 to large octagons leads to $\Delta(K) \geq A(H(K))$ (see Lemma 1.1.2). Next let $K = -K$; hence, we may assume that $H(K) = -H(K)$ (see Lemma 1.3.2). Writing $p_1, \ldots, p_6$ to denote the midpoints of the sides of $H(K)$ according to the clockwise orientation, we have $p_2 - p_1 = p_3$. Therefore, $2p_1$ and $2p_2$ generates a lattice Λ such that $\Lambda + H(K)$ is a tiling of the plane, completing the proof of Theorem 1.3.1. □

Comments. Theorem 1.3.1 is due to L. Fejes Tóth [FTL1950], and C. H. Dowker [Dow1944] proves Lemma 1.3.2 (actually for circumscribed $2m$-gons for any $m \geq 2$). According to G. Fejes Tóth and L. Fejes Tóth [FTF1973a], if a convex domain has k-fold rotational symmetry then there exists a circumscribed km-gon of minimal area that also has k-fold rotational symmetry. For a typical convex domain K, G. Fejes Tóth [FTG1995a] constructs periodic packings that are denser than any lattice packing, and his method yields that $\Delta(K) > A(H(K))$.

1.4. Packings Inside Any Convex Container

If K is a centrally symmetric convex domain then we consider packings of congruent copies of K inside any convex container. We recall that $H(K)$ denotes a circumscribed convex polygon with at most six sides of minimal area.

Theorem 1.4.1. *Given a centrally symmetric convex domain K, there exists $n(K)$ such that if a convex domain D contains $n \geq n(K)$ nonoverlapping congruent copies of K then*

$$A(D) \geq n \cdot A(H(K)).$$

Remarks. Lemma 1.1.2 and Theorem 1.3.1 show that the constant $A(H(K))$ is optimal. Moreover, the condition that n has to be large in Theorem 1.4.1 is optimal according to Example 1.4.2.

Proof. If K is a polygon of at most six sides then the hexagon bound readily holds. We verify that if K is not a parallelogram then there exist positive constants γ_1 and γ_2 depending on K such that

$$A(D) > n \cdot A(H(K)) + \gamma_1 P(D) - \gamma_2. \tag{1.6}$$

We write $K_1, \ldots, K_n$ to denote the nonoverlapping congruent copies of K in D, and we assume that D is the convex hull of these domains. Let $\Pi_1, \ldots, \Pi_n$ be the convex domains associated to $K_1, \ldots, K_n$ by Lemma 1.2.2.

In this proof σ always denote an edge of some Π_i that is contained in ∂D. Let $\sigma \subset \Pi_i$. We write x_i to denote the centre of a circular disc of radius $r(K)$ inscribed into K_i and write $C(\sigma, x_i)$ to denote the union of all segments connecting x_i to the points of σ. Since any line tangent to σ avoids K_i, we have

$$A(C(\sigma, x_i)) \geq \frac{r(K)}{2} \cdot |\sigma|, \tag{1.7}$$

where $|\cdot|$ stands for the arc length. Thus there exist positive constants λ and c_1 such that if $|\sigma| > \lambda$ then

$$A(C(\sigma, x_i)) \geq t_K(2) + c_1\,|\sigma|. \tag{1.8}$$

In addition, we define the acute angle α by the formula $\sin 3\alpha = r(K)/\lambda$; hence, if the distance of x from K_i is at most λ for some $x \notin K_i$ then the angle of the two half lines emanating from x and tangent to K_i is at least 3α.

The following considerations are based on the total curvature $\beta(\sigma)$ of σ (see Section A.5.1). Let $|\sigma| \leq \lambda$ and $\beta(\sigma) < \alpha$, and we write p and q to denote the endpoints of σ. Next we assign a compact convex set $\Omega(\sigma)$ to σ. If σ is a segment then we define $\Omega(\sigma)$ to be σ itself. If $\beta(\sigma) > 0$ then σ intersects K_i; thus, we may define $\Omega(\sigma)$ to be the triangle pqs, where the half line ps is tangent to σ, and the line sq contains the other edge of Π_i emanating from q. Since the angle of $\Omega(\sigma)$ at q is at most $\pi - 2\beta(\sigma)$ according to the definition of α, and at p is at most $\beta(\sigma)$, we deduce that

$$A(Q(\sigma)) \leq \frac{1}{2}\,\lambda^2 \sin 2\beta(\sigma) \leq c_2\,\beta(\sigma). \tag{1.9}$$

Given any Π_i, adding all $\Omega(\sigma)$ with $\sigma \subset \Pi_i$ we obtain a k_i-gon; thus,

$$A(\Pi_i) \geq t_K(k_i) + \sum_{\substack{\sigma\subset\Pi_i\\ |\sigma|>\lambda}} c_1\,|\sigma| - \sum_{\substack{\sigma\subset\Pi_i\\ |\sigma|\leq\lambda\\ \beta(\sigma)<\alpha}} c_2\,\beta(\sigma) - \sum_{\substack{\sigma\subset\Pi_i\\ |\sigma|\leq\lambda\\ \beta(\sigma)\geq\alpha}} t_K(2). \tag{1.10}$$

Now the total curvature of ∂D is 2π, which in turn yields that

$$A(D) \geq \sum_{i=1}^{n} t_K(k_i) + \sum_{|\sigma|>\lambda} c_1\,|\sigma| - c_2\,2\pi - \frac{2\pi}{\alpha}\,t_K(2). \tag{1.11}$$

It follows by the convexity of $t_K(\cdot)$ (see Corollary 1.2.4) that

$$t_K(k_i) \geq A(H(K)) + (t_K(5) - t_K(6)) \cdot (6 - k_i), \tag{1.12}$$

where $t_K(5) - t_K(6) > 0$. We write b to denote the number of edges of $\Pi_1, \ldots, \Pi_n$ that are contained in ∂D; hence, Lemma 1.2.2 (iv) leads to

$$\begin{aligned}\sum_{i=1}^{n} t_K(k_i) &\geq n \cdot A(H(K)) + (t_K(5) - t_K(6)) \cdot b\\ &\geq n \cdot A(H(K)) + \sum_{|\sigma|\leq\lambda} \frac{t_K(5) - t_K(6)}{\lambda} \cdot |\sigma|.\end{aligned}$$

In turn, we conclude (1.6) by (1.11). Now $A(D) \geq nA(K)$ and the isoperimetric inequality Theorem A.5.7 yield that $P(D) > 2\sqrt{A(K)n/\pi}$; therefore, $A(D) > n \cdot A(H(K))$ holds for large n. □

One may hope that the $n(K)$ in Theorem 1.4.1 can be chosen to be an absolute constant. We present now an example showing that this is not the case:

Example 1.4.2. Given any $c \in \mathbb{R}$, there exists a convex domain K such that any suitable $n(K)$ in Theorem 1.4.1 satisfies $n(K) \geq c$.

Let $\varepsilon > 0$ satisfy $(1/12)\ln(1/4\varepsilon) > c$. We fix a system of (x, y) coordinates, and for $p = (1, 0)$ and $q = (0, 1)$, we choose the points p_0 and q_0 in the positive corner and on the hyperbole of equation $x \cdot y = \varepsilon$ in a way that the lines pp_0 and qq_0 are tangent to the hyperbole. We write γ to denote the union of the segments pp_0 and qq_0 and the hyperbole arc between p_0 and q_0. If s is a segment of length $2(n-1)$ parallel to the first coordinate axis then $s + K$ contains n nonoverlapping translates of K, and

$$A(s+K) = 4(n-1) + A(K) = 4n - \varepsilon \cdot \ln \frac{1}{4\varepsilon} - 2\varepsilon.$$

However, the two coordinate axes and any tangent to the hyperbolic arc bound a triangle of area 2ε; hence, $A(H(K)) \geq 4 - 12\varepsilon$. Therefore, $n(K) \geq (1/12)\ln(1/4\varepsilon) > c$.

Next we investigate the structure of the optimal packing of a large number of copies.

Corollary 1.4.3. *Let K be a centrally symmetric convex domain that is not a parallelogram, and let D_n be a convex domain of minimal area that contains n nonoverlapping congruent copies of K. Then*

$$c_1\sqrt{n} < r(D_n) \leq R(D_n) < c_2\sqrt{n},$$

where $c_1, c_2 > 0$ depend only on K.

Proof. Since $H(K)$ can be assumed to be centrally symmetric according to Lemma 1.3.2, $(\sqrt{n}+2)H(K)$ contains n nonoverlapping translates of $H(K)$; hence, (1.6) yields that $P(D_n) \leq c_2\sqrt{n}$. Therefore, readily, $R(D_n) \leq c_2\sqrt{n}$, and we deduce by $r(D_n)\,P(D_n) \geq A(D_n)$ (cf. (A.8)) that $r(D_n) \geq c_1\sqrt{n}$. $\square$

Comments. Theorem 1.4.1 and Corollary 1.4.3 are proved in K. Böröczky Jr. [Bör2003b].

1.5. Noncrossing Coverings

The covering density of a convex domain K is only known in a few cases, namely, if K is a tile (hence the density is one) or K is centrally symmetric and close to being a circular disc (see the Comments to Section 1.1). Therefore, we consider a restricted class of coverings: We say that two convex domains C_1 and C_2 are *noncrossing* if there exist complementary half planes l^- and