# FINITE PACKING AND COVERING

Finite arrangements of convex bodies were intensively investigated in the second half of the twentieth century. Connections were made to many other subjects, including crystallography, the local theory of Banach spaces, and combinatorial optimization. This book, the first one dedicated solely to the subject, provides an in-depth, state-of-the-art discussion of the theory of finite packings and coverings by convex bodies. It contains various new results and arguments, besides collecting those scattered throughout the literature, and provides a comprehensive treatment of problems whose interplay was not clearly understood before. To make the material more accessible, each chapter is essentially independent, and two-dimensional and higher dimensional arrangements are discussed separately. Arrangements of congruent convex bodies in Euclidean space are discussed, and the density of finite packing and covering by balls in Euclidean, spherical, and hyperbolic spaces is considered.

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Károly Böröczky Jr.
*Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Budapest*

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CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK
40 West 20th Street, New York, NY 10011-4211, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa

http://www.cambridge.org



First published 2004

Printed in the United States of America

*Typeface* Times Roman 10.25/13 pt. *System* LaTeX $2_\varepsilon$ [TB]

*A catalog record for this book is available from the British Library.*

*Library of Congress Cataloging in Publication Data*
Böröczky, K.
Finite packing and covering / Károly Böröczky Jr.
p. cm. – (Cambridge tracts in mathematics ; 154)
Includes bibliographical references and index.
ISBN 0-521-80157-5
1. Combinatorial packing and covering. I. Title. II. Series.
QA166.7.B67 2004
511′.6 – dc22 2003065450

ISBN 0 521 80157 5 hardback

To Csilla and Csenge

# Contents

# Preface

The year 1964 witnessed the publication of two fundamental monographs about infinite packing and covering: László Fejes Tóth's *Regular Figures*, which focused on arrangements in surfaces of constant curvature, and Claude Ambrose Rogers's *Packing and Covering*, which discussed translates of a given convex body in higher dimensional Euclidean spaces. This is the finite counterpart of the story told in these works. I discuss arrangements of congruent convex bodies that either form a packing in a convex container or cover a convex shape. In the spherical and the hyperbolic space I only consider packings and coverings by balls. The most frequent quantity to be optimized is the density, which is the ratio of total volume of the congruent bodies over the volume either of the container or of the shape that is covered. In addition, extremal values of the surface area, mean width, or other fundamental quantities are also investigated in the Euclidean case. A fascinating feature of finite packings and coverings is that optimal arrangements are often related to interesting geometric shapes.

The main body of the book consists of two parts, followed by the Appendix, which discusses some important background information and prerequisites. Part 1 collects results about planar arrangements. The story starts with Farkas Bolyai and Axel Thue, who investigated specific finite packings of unit discs in the nineteenth century. After a few sporadic results, the theory of packings and coverings by copies of a convex domain started to flourish following the work of László Fejes Tóth. Here are some of the highlights: Given a convex domain $K$ in $\mathbb{R}^2$, Hugo Hadwiger proposed to find the minimal number of non-overlapping translates of $K$ touching $K$. The Hadwiger number of $K$ turns out to be eight if $K$ is a parallelogram, and six otherwise (see Theorem 2.9.1). For density-type problems, when $K$ is centrally symmetric, the hexagon bound holds for packing a high number of congruent copies of $K$ into any convex container (see Theorem 1.4.1) and for covering any convex shape by at least seven translates of $K$ (see Theorem 2.8.1). In addition, the optimal packing of $n$ unit discs has been determined for about 96% of all $n$ (see Section 4.3).

Concerning packings and coverings by equal spherical discs, the optimal arrangements have been determined when the number of discs is small, and the corresponding triangle bound has been verified with respect to any convex domain (see Sections 4.4 and 5.4). Moreover, the Euclidean triangle bound has been verified for both packings or coverings of equal circular discs with respect to large circular discs in the hyperbolic plane (see Sections 4.6 and 5.5).

Part 2 considers problems in higher dimensional spaces. For arrangements in $\mathbb{R}^d$, an important inspiration was László Fejes Tóth's celebrated Sausage Conjecture; namely, if $d \geq 5$ and the convex hull of $n$ nonoverlapping unit balls is of minimal volume then the centers are aligned, hence the convex hull is a "sausage". By now the conjecture has been verified for $d \geq 42$. Because the argument is very involved in lower dimensions, we present the proof only when the dimension is very high (see Section 8.3). Concerning mean projections, let $K$ be a convex body. If a certain mean projection of the convex hull of $n$ congruent copies of $K$ is minimal then the convex hull is approximately some ball for large $n$ (see Theorem 7.3.1). In addition, a convex compact set of maximal mean width that is covered by $n$ congruent copies of $K$ is close to some segment (see Theorem 7.4.1). Chapter 10 discusses the so-called parametric density of finite arrangements of translates of $K$, a notion due to Jörg M. Wills. For example, assuming that $K$ is centrally symmetric and the parameter $\varrho$ is not too small, the convex hull of the translates in the optimal lattice packing is essentially homothetic to a certain polytope, namely, to the so-called Wulff-shape, which is well known from crystallography.

Part 2 also discusses local translative arrangements in $\mathbb{R}^d$. The Hadwiger number is proved to be exponential in the dimension $d$ where the optimal upper bound is $3^d - 1$, and the optimal lower bound is 12 in $\mathbb{R}^3$ (see Sections 9.6 and 9.7). In addition, a family of nonoverlapping translates of $K$ that do not overlap $K$ is called a translational cloud if every half line emanating from $K$ intersects the interior of at least one translate. It has been verified that the minimal cardinality of clouds is exponential in $d^2$ (see Sections 9.12 to 9.14).

Chapter 6 of Part 2 is concerned with packing and covering by equal spherical balls on $S^d$ for $d \geq 3$. For example, the vertices of any simplicial Euclidean regular $(d+1)$-polytope determine an optimal ball packing on $S^d$. In addition, the optimal packings have been established in various cases using the so-called linear programming bound. For coverings, the optimal arrangements are known only in a very few cases, but the density estimates are very sound.

I do not discuss the arguments that use the linear programming bound in detail because related topics are thoroughly discussed in the recent monographs by J. H. Conway and N. J. A. Sloane [CoS1999], T. Ericson and V. Zinoviev [ErZ2001], and Ch. Zong [Zon1999a]. In addition to these results, many others are only surveyed at the ends of the chapters. The basic methods of proofs in this book are local volume estimates and the use of isoperimetric type inequalities. The theory of mixed volumes (the so-called Brunn–Minkowski theory) is an essential ingredient of various arguments, whereas many others are based on ingenious, rather elementary ideas. The results are not presented in their most refined form in many cases; the emphasis is rather on the underlying ideas.

I tried to keep the chapters as independent as possible; hence some ideas are repeated in a more general form in a subsequent chapter. Yet Chapter 3 and Chapter 10 rely heavily on Chapter 2 and Chapter 9, respectively. To avoid interrupting the flow of the presentation, the history of the topics is told only at the end of the sections, which is also the place to review related results.

Besides the monographs mentioned above, the interested reader may consult the forthcoming books by P. Brass, W. Moser, and J. Pach [BMP?], U. Betke, M. Henk, and J. M. Wills [BHW?], and P. Gruber [Gru?b] for in-depth discussion of many related topics.

Finally, I have to express my gratitude to many people whose help was essential in getting this project done. First, I thank my father, whose guidance from my early days led me into the Reign of Beauty and Deception called Geometry. The manuscript has been improved to a great extent by conversations with I. Bárány, U. Betke, A. Bezdek, K. Bezdek, G. Fejes Tóth, Z. Füredi, P. M. Gruber, M. Henk, A. Heppes, D. Ismailescu, W. Kuperberg, J. C. Lagarias, D. Larman, E. Makai, G. Moussong, J. Pach, T. Réti, A. Schürmann, R. Strausz, P. G. Szabó, I. Talata, T. Tarnai, Á. Tóth, G. Wintsche, and Ch. Zong. I learned the fundamental facts and methods concerning finite packings under the guidance of T. Bisztriczky and J. M. Wills. I would like to thank Béla Bollobás, Editor of the series Cambridge Tracts in Mathematics, for constant encouragement, and Roger Astley, Senior Editor at Cambridge University Press, for patiently answering all my inquiries. Finally, there are no words that express how much I owe to Csilla, who provides continuous inspiration in all aspects of my life.

# Notation

| | |
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| $\#X$: | the cardinality of $X$ if $X$ is finite |
| $\mathrm{conv}X$: | the minimal convex set containing $X$ |
| $V(\cdot)$: | volume (Lebesgue measure) |
| $\mathcal{H}^k(\cdot)$: | $k$-dimensional Hausdorff measure |
| $C + K$: | the family of $x + y$, where $x \in C$ and $y \in K$ |
| $C \ominus K$: | the family of points $x$ satisfying $x + K \subset C$ |
| $L^\perp$: | the orthogonal complement of the affine subspace $L$ |
| $u^\perp$: | the family of vectors orthogonal to the nonzero vector $u$ |
| $X\|L$: | the orthogonal projection of a set $X$ into $L$ |
| $X/L$: | the orthogonal projection of a set $X$ into $L^\perp$ |
| $\mathrm{aff}X$: | the minimal affine subspace containing a set $X$ |
| $\chi_K(\cdot)$: | the characteristic function of a set $K$ |
| $\partial K$: | the relative boundary of a compact convex set $K$ |
| $h_K(\cdot)$: | the support function of a compact convex set $K$ |
| $\lvert K\rvert$: | the relative content of a compact convex set $K$ |
| $\dim K$: | the dimension of the affine hull of a compact convex set $K$ |
| $\delta(K)$: | the packing density of a convex body $K$ |
| $\delta_T(K)$: | the translative packing density of a convex body $K$ |
| $\vartheta(K)$: | the covering density of a convex body $K$ |
| $\vartheta_T(K)$: | the translative covering density of a convex body $K$ |
| $\lVert\cdot\rVert_K$: | the norm with respect to the $o$-symmetric convex body $K$ |
| $\langle\cdot,\cdot\rangle$: | scalar product |
| $\lVert\cdot\rVert$: | Euclidean norm |
| $\det\Lambda$: | the determinant of a lattice $\Lambda$ |
| $B^d$: | the Euclidean unit ball in $\mathbb{R}^d$ |
| $\kappa_d$: | $V(B^d)$ |
| $S^d$: | the $d$-sphere |
| $H^d$: | hyperbolic $d$-space |
| $\mathbb{S}/\Lambda$: | the quotient of the space $\mathbb{S}$ with respect to a lattice $\Lambda$ |
| $\lfloor x\rfloor$: | the largest integer that is at most $x$ |
| $\lceil x\rceil$: | the smallest integer that is at least $x$ |