PART I

SPIN-1/2 FERMIONS IN QUANTUM FIELD THEORY, THE STANDARD MODEL, AND BEYOND

1 Two-Component Formalism for Spin-1/2 Fermions

In this chapter, we examine the incorporation of spin-1/2 fermions into quantum field theory. Underlying the relativistic theory of quantized fields is special relativity and the invariance of the Lagrangian under the Poincaré group, which comprise Lorentz transformations and spacetime translations (e.g., see [B130–B132]).

1.1 The Lorentz Group and Its Lie Algebra

Under an *active* Lorentz transformation, $\Lambda^\mu{}_\nu$, the spacetime coordinates, x^μ, transform as $x'^\mu = \Lambda^\mu{}_\nu x^\nu$, where there is an implicit sum over the repeated index ν (see Appendix A.4 for our conventions for spacetime notation). One can regard $\Lambda^\mu{}_\nu$ as the matrix element located in row μ and column ν of the 4×4 matrix Λ. The condition that $g_{\mu\nu}x^\mu x^\nu$ is invariant under Lorentz transformations implies that

$$\Lambda^\mu{}_\nu g_{\mu\rho} \Lambda^\rho{}_\lambda = g_{\lambda\nu}\,. \tag{1.1}$$

The set of all Λ that satisfy eq. (1.1) forms a Lie group O(1,3).

Equation (1.1) implies that Λ has the following two properties: (i) $\det \Lambda = \pm 1$ and (ii) $|\Lambda^0{}_0| \geq 1$. Thus, Lorentz transformations fall into four disconnected classes, which can be denoted by a pair of signs: $\big(\mathrm{sgn}(\det\Lambda)\,,\,\mathrm{sgn}(\Lambda^0{}_0)\big)$. The proper orthochronous Lorentz transformations correspond to $(+,+)$ and are continuously connected to the identity. The group of such transformations forms a subgroup of O(1,3), which we shall denote by $\mathrm{SO}_+(1{,}3)$.[1] If $\Lambda \in \mathrm{SO}_+(1,3)$, we can generate all elements of the other classes of Lorentz transformations by introducing the space-inversion matrix, $\Lambda_P = \mathrm{diag}(1,-1,-1,-1)$, the time-inversion matrix $\Lambda_T = \mathrm{diag}(-1,1,1,1)$ and the spacetime-inversion matrix $\Lambda_P\Lambda_T = -\mathbb{1}_{4\times 4}$. Then, $\{\Lambda\,,\,\Lambda_P\Lambda\,,\,\Lambda_T\Lambda\,,\,\Lambda_P\Lambda_T\Lambda\}$ spans the full Lorentz group.

Infinitesimal Lorentz transformations must be proper and orthochronous, since these are continuously connected to the identity. These transformations are best studied by exploring the properties of the SO(1,3) Lie algebra, denoted henceforth by $\mathfrak{so}(1,3)$. Later (see Section 1.7), we shall examine the implications of the improper Lorentz transformations: space inversion and time inversion.

The most general proper orthochronous Lorentz transformation matrix Λ is characterized by a rotation angle θ about an axis $\boldsymbol{\hat{n}}$ $[\vec{\boldsymbol{\theta}} \equiv \theta\boldsymbol{\hat{n}}]$ and a boost vector

[1] In this notation, the S (which stands for "special") corresponds to the condition $\det\Lambda = 1$ and the subscript + corresponds to $\Lambda^0{}_0 \geq +1$.

$\vec{\boldsymbol{\zeta}} \equiv \hat{\boldsymbol{v}} \tanh^{-1}\beta$ [where $\hat{\boldsymbol{v}} \equiv \vec{\boldsymbol{v}}/|\vec{\boldsymbol{v}}|$ is the unit velocity vector and $\beta \equiv |\vec{\boldsymbol{v}}|/c$].[2] The 4×4 matrix Λ is given by[3]

$$\Lambda = \exp\left(-\tfrac{1}{2}i\theta_{\rho\sigma}s^{\rho\sigma}\right) = \exp\left(-i\vec{\boldsymbol{\theta}}\boldsymbol{\cdot}\vec{\boldsymbol{s}} - i\vec{\boldsymbol{\zeta}}\boldsymbol{\cdot}\vec{\boldsymbol{k}}\right) = \exp\begin{pmatrix} 0 & \zeta^1 & \zeta^2 & \zeta^3 \\ \zeta^1 & 0 & -\theta^3 & \theta^2 \\ \zeta^2 & \theta^3 & 0 & -\theta^1 \\ \zeta^3 & -\theta^2 & \theta^1 & 0 \end{pmatrix}, \tag{1.2}$$

where $\theta_{\rho\sigma} = -\theta_{\sigma\rho}$, and the $s^{\rho\sigma} = -s^{\sigma\rho}$ are six independent 4×4 antisymmetric matrices that satisfy the commutation relations of the Lie algebra $\mathfrak{so}(1,3) \simeq \mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$,[4]

$$[s^{\alpha\beta}, s^{\rho\sigma}] = i(g^{\beta\rho}\, s^{\alpha\sigma} - g^{\alpha\rho}\, s^{\beta\sigma} - g^{\beta\sigma}\, s^{\alpha\rho} + g^{\alpha\sigma}\, s^{\beta\rho}). \tag{1.3}$$

The matrix elements of $s^{\rho\sigma}$ are given explicitly as follows (e.g., see [B133]):

$$(s^{\rho\sigma})^{\mu}{}_{\nu} = i(\delta^{\sigma}_{\nu}\, g^{\mu\rho} - \delta^{\rho}_{\nu}\, g^{\mu\sigma})\,. \tag{1.4}$$

In eq. (1.2), we have also defined $\theta^i \equiv \frac{1}{2}\epsilon^{ijk}\theta^{jk}$, $\zeta^i \equiv \theta^{i0} = -\theta^{0i}$, $s^i \equiv \frac{1}{2}\epsilon^{ijk}s^{jk}$, and $k^i \equiv s^{0i} = -s^{i0}$, after noting that $\theta_{ij} = \theta^{ij}$ and $\theta_{0i} = -\theta^{0i}$. Here, the indices $i,j,k \in \{1,2,3\}$ and $\epsilon^{123} = +1$.

In light of eqs. (1.2) and (1.4), an infinitesimal orthochronous Lorentz transformation is given by

$$\Lambda^{\mu}{}_{\nu} \simeq \delta^{\mu}_{\nu} - \tfrac{1}{2}i\theta_{\rho\sigma}(s^{\rho\sigma})^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \tfrac{1}{2}\theta_{\rho\sigma}(\delta^{\sigma}_{\nu}\, g^{\mu\rho} - \delta^{\rho}_{\nu}\, g^{\mu\sigma}) = \delta^{\mu}_{\nu} + \tfrac{1}{2}(\theta^{\mu}{}_{\nu} - \theta_{\nu}{}^{\mu})\,. \tag{1.5}$$

Since $\theta^{\mu}{}_{\nu} = g^{\mu\rho}\theta_{\rho\nu} = -g^{\mu\rho}\theta_{\nu\rho} = -\theta_{\nu}{}^{\mu}$, it follows that

$$\Lambda^{\mu}{}_{\nu} \simeq \delta^{\mu}_{\nu} + \theta^{\mu}{}_{\nu} \simeq (\mathbb{1}_{4\times 4} - i\vec{\boldsymbol{\theta}}\boldsymbol{\cdot}\vec{\boldsymbol{s}} - i\vec{\boldsymbol{\zeta}}\boldsymbol{\cdot}\vec{\boldsymbol{k}})^{\mu}{}_{\nu}\,, \tag{1.6}$$

where $\mathbb{1}_{4\times 4}$ is the 4×4 identity matrix. That is, the s^i generate infinitesimal three-dimensional rotations in space and the k^i generate infinitesimal Lorentz boosts.

The spin-0 and (massive) spin-1 fields transform under a general proper orthochronous Lorentz transformation as[5]

$$\phi'(x') = \phi(x)\,, \qquad \text{spin } 0\,, \tag{1.7}$$

$$A'^{\mu}(x') = \Lambda^{\mu}{}_{\nu}A^{\nu}(x)\,, \qquad \text{spin } 1\,. \tag{1.8}$$

For a field of spin s, the general transformation law reads

$$\psi'_{\alpha}(x') = \exp\left(-\tfrac{1}{2}i\theta_{\mu\nu}S^{\mu\nu}\right)_{\alpha}{}^{\beta}\, \psi_{\beta}(x)\,, \tag{1.9}$$

[2] Henceforth, we work in units where $\hbar = c = 1$ (see Appendix A.3).

[3] All symmetry transformations in this chapter are defined from the active point of view. For a passive Lorentz transformation, where the coordinate frame is transformed and the four-vectors are held fixed, simply replace $\{\vec{\boldsymbol{\theta}}, \vec{\boldsymbol{\zeta}}\}$ with $\{-\vec{\boldsymbol{\theta}}, -\vec{\boldsymbol{\zeta}}\}$.

[4] If $\{\boldsymbol{t_1}, \boldsymbol{t_2}, \boldsymbol{t_3}\}$ is a basis for the complex Lie algebra $\mathfrak{sl}(2,\mathbb{C})$, then the notation $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$ corresponds to the realification of $\mathfrak{sl}(2,\mathbb{C})$, which yields a real Lie algebra consisting of real linear combinations of the six generators, $\{\boldsymbol{t_1}, i\boldsymbol{t_1}, \boldsymbol{t_2}, i\boldsymbol{t_2}, \boldsymbol{t_3}, i\boldsymbol{t_3}\}$.

[5] Equations (1.7) and (1.8) can also be written as $\phi'(x) = \phi(\Lambda^{-1}x)$ and $A'^{\mu}(x) = \Lambda^{\mu}{}_{\nu}A^{\nu}(\Lambda^{-1}x)$. The transformation law for a massless spin-1 gauge field is more complicated and has the form $A'^{\mu}(x) + \partial^{\mu}\Omega(x,\Lambda) = \Lambda^{\mu}{}_{\nu}A^{\nu}(\Lambda^{-1}x)$, as indicated in eq. (5.9.31) of [B72], since $A'^{\mu}(x)$ and the gauge-transformed $A'^{\mu}(x) + \partial^{\mu}\Omega(x,\Lambda)$ are physically equivalent.

where the $S^{\mu\nu}$ are (finite-dimensional) irreducible matrix representations of the Lie algebra of the Lorentz group, and α and β label the components of the matrix representation space. The dimension of this space is related to the spin of the particle. In particular, $S^{\mu\nu}$ is an antisymmetric tensor, $S^{\mu\nu} = -S^{\nu\mu}$, that satisfies the commutation relations of $\mathfrak{so}(1,3)$ [eq. (1.3)]. Different irreducible finite-dimensional representations of $\mathfrak{so}(1,3)$ correspond to particles of different spin. In analogy with s^i and k^i defined below eq. (1.4), we identify the following pieces of $S^{\mu\nu}$:

$$S^i \equiv \tfrac{1}{2}\epsilon^{ijk}S^{jk}\,, \qquad\qquad K^i \equiv S^{0i}\,, \tag{1.10}$$

where $i, j, k \in \{1, 2, 3\}$. Using eq. (1.3), it follows that S^i and K^i satisfy the commutation relations

$$[S^i, S^j] = i\epsilon^{ijk}S^k\,, \tag{1.11}$$

$$[S^i, K^j] = i\epsilon^{ijk}K^k\,, \tag{1.12}$$

$$[K^i, K^j] = -i\epsilon^{ijk}S^k\,. \tag{1.13}$$

It is convenient to define the following linear combinations of the generators:

$$\vec{\boldsymbol{S}}_+ \equiv \tfrac{1}{2}(\vec{\boldsymbol{S}} + i\vec{\boldsymbol{K}})\,, \tag{1.14}$$

$$\vec{\boldsymbol{S}}_- \equiv \tfrac{1}{2}(\vec{\boldsymbol{S}} - i\vec{\boldsymbol{K}})\,. \tag{1.15}$$

Then, eqs. (1.11)–(1.13) decouple and yield two independent $\mathfrak{su}(2)$ Lie algebras:

$$[S^i_+, S^j_+] = i\epsilon^{ijk}S^k_+\,, \tag{1.16}$$

$$[S^i_-, S^j_-] = i\epsilon^{ijk}S^k_-\,, \tag{1.17}$$

$$[S^i_+, S^j_-] = 0\,. \tag{1.18}$$

The finite-dimensional irreducible representations of $\mathfrak{su}(2)$ are well known: these are the $(2s + 1) \times (2s + 1)$ representation matrices corresponding to spin s, where $s = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ (whose matrix elements appear in most textbooks on quantum mechanics; e.g., see [B134]). Hence, the irreducible representations of the Lorentz group can be characterized by two numbers (s_+, s_-), where $s_\pm$ is nonnegative and either an integer or a half-integer. The eigenvalues of $\vec{\boldsymbol{S}}^{\,2}_\pm$ are given by $s_\pm(s_\pm + 1)$. The dimension of the representation corresponding to (s_+, s_-) is $(2s_+ +1)(2s_- +1)$.

Using eqs. (1.9) and (1.10), an infinitesimal Lorentz transformation is given by

$$M \equiv \exp\left(-\tfrac{1}{2}i\theta_{\mu\nu}S^{\mu\nu}\right) \simeq \mathbb{1} - i\vec{\boldsymbol{\theta}}\boldsymbol{\cdot}\vec{\boldsymbol{S}} - i\vec{\boldsymbol{\zeta}}\boldsymbol{\cdot}\vec{\boldsymbol{K}}\,, \tag{1.19}$$

where θ^i and ζ^i are defined following eq. (1.2) and $\mathbb{1}$ is the identity. The simplest (trivial) representation is the one-dimensional $(0, 0)$ representation, which corresponds to a spin-0 scalar field. In this representation, $\vec{\boldsymbol{S}} = \vec{\boldsymbol{K}} = 0$ and we recover from eq. (1.9) the transformation law given in eq. (1.7). The spin-1 transformation law [see eq. (1.8)] corresponds to the four-dimensional $(\frac{1}{2}, \frac{1}{2})$ representation. However, in a quantum field theory of massive spin-1 fields, only three of the four degrees of freedom are physical. Moreover, in gauge theories of massless spin-1 fields, gauge invariance introduces an additional constraint and only two degrees of freedom are physical. This is described in detail in [B72], to which we refer the reader.

1.2 The Poincaré Group and Its Lie Algebra

The Poincaré group is a semidirect product of the group of spacetime translations and the Lorentz group (e.g., see [B112, B132]). In particular, under a Poincaré transformation, the spacetime coordinates transform as $x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu}$, where Λ satisfies eq. (1.1) and a^{μ} is a constant four-vector. In examining the behavior of the fields under a Poincaré transformation, we shall assume that Λ is a proper orthochronous Lorentz transformation [see eq. (1.2)].

Under a Poincaré transformation, a field of spin s transforms according to eq. (1.9) after identifying x' with the Poincaré transformed spacetime coordinate. It is convenient to rewrite this transformation law as follows:

$$\psi'_{\alpha}(x) = \exp\big(-\tfrac{1}{2} i\theta_{\mu\nu} S^{\mu\nu}\big)_{\alpha}{}^{\beta}\, \psi_{\beta}\left(\Lambda^{-1}(x-a)\right) , \tag{1.20}$$

where we have used $x = \Lambda^{-1}(x' - a)$ and redefined the dummy variable x' by removing the prime. Under an infinitesimal Poincaré transformation, we expand eq. (1.20) about $\Lambda = \mathbb{1}_{4\times 4}$ and $a = 0$ to obtain[6]

$$\psi'_{\alpha}(x) \simeq \left[\mathbb{1} + i a_{\mu} P^{\mu} - \tfrac{1}{2} i\theta_{\mu\nu}(L^{\mu\nu} + S^{\mu\nu})\right]_{\alpha}{}^{\beta}\, \psi_{\beta}(x)\,, \tag{1.21}$$

where P^{μ} and $L^{\mu\nu}$ are the differential operators[7]

$$P^{\mu} \equiv i\partial^{\mu}\,, \qquad L^{\mu\nu} \equiv i(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu})\,, \tag{1.22}$$

and the total angular momentum tensor is $J^{\mu\nu} \equiv L^{\mu\nu} + S^{\mu\nu}$. The four generators of spacetime translations (P^{μ}) and the six generators of Lorentz transformations ($J^{\mu\nu}$, $\mu < \nu$) satisfy the following commutation relations of the Poincaré algebra:[8]

$$[P^{\mu}, P^{\nu}] = 0\,, \tag{1.23}$$

$$[J^{\mu\nu}, P^{\rho}] = i(g^{\nu\rho} P^{\mu} - g^{\mu\rho} P^{\nu})\,, \tag{1.24}$$

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(g^{\nu\rho}\, J^{\mu\sigma} - g^{\mu\rho}\, J^{\nu\sigma} - g^{\nu\sigma}\, J^{\mu\rho} + g^{\mu\sigma}\, J^{\nu\rho})\,. \tag{1.25}$$

It is convenient to introduce the Pauli–Lubański vector w^{μ}:

$$w^{\mu} \equiv -\tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma} J_{\nu\rho} P_{\sigma} = (\vec{\boldsymbol{J}}\boldsymbol{\cdot}\vec{\boldsymbol{P}}\,;\, P^0 \vec{\boldsymbol{J}} + \vec{\boldsymbol{\mathcal{K}}} \times \vec{\boldsymbol{P}})\,, \tag{1.26}$$

in a convention where $\epsilon^{0123} = 1$, where $J^i \equiv \frac{1}{2}\epsilon^{ijk} J^{jk}$ and $\mathcal{K}^i \equiv J^{0i}$. It follows that $w_{\mu}P^{\mu} = 0$. The Poincaré algebra possesses two independent Casimir operators $P^2 \equiv P_{\mu}P^{\mu}$ and $w^2 \equiv w_{\mu}w^{\mu}$, which commute with the generators P^{μ} and $J^{\mu\nu}$.

The representations of the Poincaré group, which correspond to particle states of nonnegative energy P^0 with definite mass and spin, can be labeled by the eigenvalues of the Casimir operators, P^2 and w^2, when acting on the physical states.

[6] The operators $\mathbb{1}$, P^{μ}, and $L^{\mu\nu}$ include an implicit factor of $\delta_{\alpha}{}^{\beta}$, whereas the spin operator $S^{\mu\nu}$ depends nontrivially on α and β (except for the case of spin 0, where $S = 0$).

[7] We recognize P^{μ} and $L^{\mu\nu}$ as the quantum mechanical four-vector momentum operator and tensor orbital angular momentum operator, respectively, in the coordinate representation.

[8] As demonstrated in Exercise 1.2, eq. (1.24) is a consequence of the transformation law of the four-vector P^{μ} under Lorentz transformations.

The eigenvalue of P^2 is m^2, where m is the mass. To see the physical interpretation of w^2, we first consider the case of $m \neq 0$. In this case, we are free to evaluate w^2 in the particle rest frame (since w^2 is a Lorentz scalar). In this frame, $w^\mu = (0\,; m\vec{\boldsymbol{S}})$, where S^i is defined in eq. (1.10). Hence, $w^2 = -m^2\vec{\boldsymbol{S}}^{\,2}$, with eigenvalues $-m^2 s(s+1)$ [where $s = 0, \frac{1}{2}, 1, \ldots$]. We conclude that massive (positive-energy) states can be labeled by (m, s), where m is the mass and s is the spin of the state.

If $m = 0$, the previous analysis is not valid, since we cannot evaluate w^2 in the rest frame. Nevertheless, if we take the $m \to 0$ limit, it follows from the results above that either $w^2 = 0$ or the corresponding states have infinite spin. We reject the second possibility (which does not appear to be realized in Nature) and assume that $w^2 = 0$. Thus, we must solve the equations $w^2 = P^2 = w_\mu P^\mu = 0$. It is simplest to choose a frame in which $P = (P^0; 0, 0, P^0)$ where $P^0 > 0$. In this frame, it is easy to show that $w = (w^0; 0, 0, w^0)$. That is, in any Lorentz frame,

$$w^\mu = hP^\mu\,, \tag{1.27}$$

where h is called the helicity operator. In particular,

$$[h\,,\, P^\mu] = [h\,,\, J^{\mu\nu}] = 0\,, \tag{1.28}$$

which means that the eigenvalues of h can be used to label states of the irreducible massless representations of the Poincaré algebra. From eq. (1.27), we derive[9]

$$h = \frac{w^0}{P^0} = \frac{\vec{\boldsymbol{J}}\boldsymbol{\cdot}\vec{\boldsymbol{P}}}{P^0} = \frac{\vec{\boldsymbol{S}}\boldsymbol{\cdot}\vec{\boldsymbol{P}}}{P^0}\,. \tag{1.29}$$

Eigenvalues of h are called the helicity (and are denoted by λ). In particular, note that for massless states, the eigenvalue of $\vec{\boldsymbol{S}}\boldsymbol{\cdot}\vec{\boldsymbol{P}}/P^0$ is equal to that of $\vec{\boldsymbol{S}}\boldsymbol{\cdot}\hat{\boldsymbol{P}}$ (where $\hat{\boldsymbol{P}} \equiv \vec{\boldsymbol{P}}/|\vec{\boldsymbol{P}}|$). Moreover, $\vec{\boldsymbol{S}}\boldsymbol{\cdot}\hat{\boldsymbol{P}}$ corresponds to the projection of the spin along its direction of motion with a spectrum consisting of $\lambda = 0, \pm\frac{1}{2}, \pm 1, \ldots$, where $\lambda \to -\lambda$ under a CPT transformation. Thus, in any quantum field theory realization of massless particles, both $\pm|\lambda|$ helicity states must appear in the theory. It is common to refer to a massless (positive-energy) state of helicity $|\lambda|$ as having spin-$|\lambda|$.

1.3 Spin-1/2 Representation of the Lorentz Group

We first focus on the simplest nontrivial irreducible representations of the Lorentz algebra: the two-dimensional representations $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$. The corresponding two-dimensional representations of the Lorentz generators are explicitly given by

$$(\tfrac{1}{2}, 0):\quad \vec{\boldsymbol{S}}_+ = \tfrac{1}{2}(\vec{\boldsymbol{S}} + i\vec{\boldsymbol{K}}) = \tfrac{1}{2}\vec{\boldsymbol{\sigma}}\,, \qquad \vec{\boldsymbol{S}}_- = \tfrac{1}{2}(\vec{\boldsymbol{S}} - i\vec{\boldsymbol{K}}) = 0\,, \tag{1.30}$$

which corresponds to $\vec{\boldsymbol{S}} = \vec{\boldsymbol{\sigma}}/2$ and $\vec{\boldsymbol{K}} = -i\vec{\boldsymbol{\sigma}}/2$, and

$$(0, \tfrac{1}{2}):\quad \vec{\boldsymbol{S}}_+ = \tfrac{1}{2}(\vec{\boldsymbol{S}} + i\vec{\boldsymbol{K}}) = 0\,, \qquad \vec{\boldsymbol{S}}_- = \tfrac{1}{2}(\vec{\boldsymbol{S}} - i\vec{\boldsymbol{K}}) = \tfrac{1}{2}\vec{\boldsymbol{\sigma}}\,. \tag{1.31}$$

[9] The three-vector orbital angular momentum operator is given by $L^i \equiv \frac{1}{2}\epsilon^{ijk}L^{jk}$ [see eq. (1.22)]. Hence, $\vec{\boldsymbol{L}} = \vec{\boldsymbol{x}} \times \vec{\boldsymbol{P}}$ and it follows that $\vec{\boldsymbol{J}}\boldsymbol{\cdot}\vec{\boldsymbol{P}} = (\vec{\boldsymbol{L}} + \vec{\boldsymbol{S}})\boldsymbol{\cdot}\vec{\boldsymbol{P}} = (\vec{\boldsymbol{x}} \times \vec{\boldsymbol{P}} + \vec{\boldsymbol{S}})\boldsymbol{\cdot}\vec{\boldsymbol{P}} = \vec{\boldsymbol{S}}\boldsymbol{\cdot}\vec{\boldsymbol{P}}$.

Hence, we can identify $\vec{S} = \vec{\sigma}/2$ and $\vec{K} = i\vec{\sigma}/2$. Here, $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ are the usual Pauli spin matrices,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.32}$$

It is convenient to define a fourth Pauli matrix,

$$\sigma^0 = \mathbb{1}_{2\times 2}, \tag{1.33}$$

where $\mathbb{1}_{2\times 2}$ is the 2×2 identity matrix. We can then define the four Pauli matrices in a unified notation, $\sigma^\mu = (\mathbb{1}_{2\times 2}\,;\,\vec{\sigma})$.[10]

Consider the infinitesimal Lorentz transformation in the $(\frac{1}{2}, 0)$ representation. Inserting the $(\frac{1}{2}, 0)$ generators [eq. (1.30)] into eq. (1.19) yields

$$M = \exp\left(-\tfrac{1}{2} i\vec{\theta}\boldsymbol{\cdot}\vec{\sigma} - \tfrac{1}{2}\vec{\zeta}\boldsymbol{\cdot}\vec{\sigma}\right) \simeq \mathbb{1}_{2\times 2} - \tfrac{1}{2} i\vec{\theta}\boldsymbol{\cdot}\vec{\sigma} - \tfrac{1}{2}\vec{\zeta}\boldsymbol{\cdot}\vec{\sigma}. \tag{1.34}$$

A two-component $(\frac{1}{2}, 0)$ spinor field is denoted by $\chi_\alpha(x)$, and transforms as

$$\chi'_\alpha(x') = M_\alpha{}^\beta \chi_\beta(x), \qquad \alpha, \beta \in \{1, 2\}. \tag{1.35}$$

By definition, M carries undotted spinor indices, as indicated by $M_\alpha{}^\beta$. In our conventions for the location of the spinor indices, we sum implicitly over a repeated index pair in which one index is lowered and one index is raised.

If M is a matrix representation of SL(n,$\mathbb{C}$), then M^*, $(M^{-1})^{\mathsf{T}}$ and $(M^{-1})^\dagger$ are also matrix representations of the same dimension. For $n > 2$, all four representations are inequivalent. For SL(2,$\mathbb{C}$), there are at most two distinct matrix representations corresponding to a given dimension: (j_1, j_2) and (j_2, j_1). Using eq. (1.34) and the following property of Pauli matrices,

$$\sigma^2 \vec{\sigma} (\sigma^2)^{\mathsf{T}} = \vec{\sigma}^{\mathsf{T}}, \tag{1.36}$$

where the transpose of the σ-matrices are $\vec{\sigma}^{\mathsf{T}} = (\sigma^1, -\sigma^2, \sigma^3)$, it follows that M and $(M^{-1})^{\mathsf{T}}$ are related by

$$(M^{-1})^{\mathsf{T}} = i\sigma^2 M (i\sigma^2)^{\mathsf{T}}. \tag{1.37}$$

Since $(i\sigma^2)^{\mathsf{T}} = (i\sigma^2)^{-1}$, the matrices M and $(M^{-1})^{\mathsf{T}}$ are related by a similarity transformation, corresponding to a unitary change in basis. Hence, M and $(M^{-1})^{\mathsf{T}}$ are equivalent representations.[11]

It is convenient to introduce the two-component spinor field $\chi^\alpha(x)$, which under the contragredient representation $(M^{-1})^{\mathsf{T}}$ transforms as

$$\chi'^\alpha(x') = [(M^{-1})^{\mathsf{T}}]^\alpha{}_\beta\, \chi^\beta(x) = [i\sigma^2 M (i\sigma^2)^{\mathsf{T}}]^\alpha{}_\beta\, \chi^\beta(x). \tag{1.38}$$

[10] The beginning student often misinterprets the symbol σ^2 to mean the square of σ. In the notation of eqs. (1.32) and (1.33), the superscripts are analogous to contravariant indices. Although the σ^μ do not transform under Lorentz transformations, we will see shortly that one can create contravariant four-vectors by suitably employing σ^μ [see eq. (1.102)].

[11] This corresponds to the well-known result that the **2** and $\overline{\mathbf{2}}$ representations of SU(2) are equivalent.

This motivates the definitions

$$\epsilon^{\alpha\beta} \equiv i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \epsilon_{\alpha\beta} \equiv (i\sigma^2)^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{1.39}$$

In particular, the nonzero components of the epsilon symbols are

$$\epsilon^{12} = -\epsilon^{21} = \epsilon_{21} = -\epsilon_{12} = 1. \tag{1.40}$$

We also introduce the two-index symmetric Kronecker delta symbol,

$$\delta_1^1 = \delta_2^2 = 1\,, \qquad \delta_2^1 = \delta_2^1 = 0\,. \tag{1.41}$$

The epsilon symbols satisfy

$$\epsilon_{\alpha\beta}\epsilon^{\rho\tau} = -\delta_\alpha{}^\rho\delta_\beta{}^\tau + \delta_\alpha{}^\tau\delta_\beta{}^\rho\,, \tag{1.42}$$

from which it follows that[12]

$$\epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = \delta_\alpha{}^\gamma\,, \qquad \epsilon^{\gamma\beta}\epsilon_{\beta\alpha} = \delta^\gamma{}_\alpha\,. \tag{1.43}$$

Finally, the following equation, often called the Schouten identity, is noteworthy:

$$\epsilon_{\alpha\beta}\epsilon_{\gamma\delta} + \epsilon_{\alpha\gamma}\epsilon_{\delta\beta} + \epsilon_{\alpha\delta}\epsilon_{\beta\gamma} = 0\,. \tag{1.44}$$

Equations (1.35) and (1.38) imply that

$$\chi^\alpha = \epsilon^{\alpha\beta}\chi_\beta\,, \qquad \chi_\alpha = \epsilon_{\alpha\beta}\chi^\beta\,. \tag{1.45}$$

That is, the epsilon symbols can be used to raise or lower a spinor index. In particular, in raising or lowering an index of a spinor quantity, *adjacent* spinor indices are summed over when multiplied on the *left* by the appropriate epsilon symbol. As noted below eq. (1.37), M and $(M^{-1})^{\mathsf{T}}$ are equivalent representations. Hence, χ_α and χ^α are equally good candidates for the $(\frac{1}{2}, 0)$ representation.

Consider next the infinitesimal Lorentz transformation in the $(0, \frac{1}{2})$ representation [eqs. (1.19) and (1.31)]:

$$(M^{-1})^\dagger = \exp\left(-\tfrac{1}{2}i\vec{\boldsymbol{\theta}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}} + \tfrac{1}{2}\vec{\boldsymbol{\zeta}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\right) \simeq \mathbb{1}_{2\times 2} - \tfrac{1}{2}i\vec{\boldsymbol{\theta}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}} + \tfrac{1}{2}\vec{\boldsymbol{\zeta}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\,, \tag{1.46}$$

after taking the inverse of the hermitian conjugate of eq. (1.34). A two-component $(0, \frac{1}{2})$ spinor field is denoted by $\eta^{\dagger\,\dot\alpha}(x)$, and transforms as

$$\eta'^{\,\dagger\,\dot\alpha}(x') = (M^{-1})^{\dagger\,\dot\alpha}{}_{\dot\beta}\,\eta^{\dagger\,\dot\beta}(x)\,, \qquad \dot\alpha, \dot\beta \in \{\dot 1, \dot 2\}. \tag{1.47}$$

By definition, $(M^{-1})^\dagger$ carries dotted spinor indices, as indicated by $(M^{-1})^{\dagger\,\dot\alpha}{}_{\dot\beta}$. Here, the "dotted" indices have been introduced to distinguish the $(0, \frac{1}{2})$ representation from the $(\frac{1}{2}, 0)$ representation.

The equivalent description of this representation is obtained via the conjugate representation M^*. Taking the complex conjugate of eq. (1.37), it follows that M^*

[12] In light of eq. (1.41), the distinction between $\delta_\alpha{}^\gamma$ and $\delta^\gamma{}_\alpha$ is somewhat pedantic. Nevertheless, it is useful to keep track of this distinction when reinterpreting such equations in terms of matrix multiplication.

is related by a similarity transformation to $(M^{-1})^\dagger$. The two-component spinor field $\eta^\dagger_{\dot\alpha}(x)$ under the conjugate representation transforms as

$$\eta'^{\dagger}_{\dot\alpha}(x') = (M^*)_{\dot\alpha}{}^{\dot\beta}\eta^\dagger_{\dot\beta}(x), \tag{1.48}$$

where

$$\eta^\dagger_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\eta^{\dagger\,\dot\beta}\,, \qquad\qquad \eta^{\dagger\,\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\eta^\dagger_{\dot\beta}, \tag{1.49}$$

and

$$\epsilon^{\dot\alpha\dot\beta} \equiv (\epsilon^{\alpha\beta})^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \epsilon_{\dot\alpha\dot\beta} \equiv (\epsilon_{\alpha\beta})^* = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{1.50}$$

Hence, $\epsilon^{\dot\alpha\dot\beta}$ and $\epsilon^{\alpha\beta}$ are numerically equal, but the dotted and undotted indices transform differently under Lorentz transformations [see eqs. (1.35) and (1.47)] and must always be kept distinct. Likewise, we define the Kronecker delta symbol with dotted indices, $\delta^{\dot\beta}_{\dot\alpha} \equiv (\delta^\beta_\alpha)^*$.

The dotted epsilon tensor satisfies

$$\epsilon_{\dot\alpha\dot\beta}\epsilon^{\dot\rho\dot\tau} = -\delta_{\dot\alpha}{}^{\dot\rho}\delta_{\dot\beta}{}^{\dot\tau} + \delta_{\dot\alpha}{}^{\dot\tau}\delta_{\dot\beta}{}^{\dot\rho}\,, \tag{1.51}$$

from which it follows that

$$\epsilon_{\dot\alpha\dot\beta}\epsilon^{\dot\beta\dot\gamma} = \delta_{\dot\alpha}{}^{\dot\gamma}\,, \qquad\qquad \epsilon^{\dot\gamma\dot\beta}\epsilon_{\dot\beta\dot\alpha} = \delta^{\dot\gamma}{}_{\dot\alpha}\,. \tag{1.52}$$

Likewise, the Schouten identity analogous to that of eq. (1.44) also holds:

$$\epsilon_{\dot\alpha\dot\beta}\epsilon_{\dot\gamma\dot\delta} + \epsilon_{\dot\alpha\dot\gamma}\epsilon_{\dot\delta\dot\beta} + \epsilon_{\dot\alpha\dot\delta}\epsilon_{\dot\beta\dot\gamma} = 0\,. \tag{1.53}$$

Note that the $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ representations are related by conjugation. That is, if ψ_α is a $(\frac{1}{2},0)$ fermion, then $(\psi_\alpha)^\dagger$ transforms as a $(0,\frac{1}{2})$ fermion. In this context, the conjugation symbol $(^\dagger)$ denotes complex conjugation for classical fields or hermitian conjugation for quantum field operators. In particular, we shall identify[13]

$$\psi^\dagger_{\dot\alpha} \equiv (\psi_\alpha)^\dagger. \tag{1.54}$$

This means that we can, and will, describe all fermion degrees of freedom using only fields defined as left-handed $(\frac{1}{2},0)$ fermions ψ_α, and their conjugates. In combining spinors to make Lorentz tensors (as in Section 1.4), it is useful to regard $\psi^\dagger_{\dot\alpha}$ as a row vector, and ψ_α as a column vector. Likewise, it follows that $\psi_\alpha = (\psi^\dagger_{\dot\alpha})^\dagger$. The Lorentz transformation property of $\eta^\dagger_{\dot\alpha}$ then follows from eq. (1.48), which can be rewritten as $[\eta_\alpha(x)]^\dagger \to [\eta_\beta(x)]^\dagger (M^\dagger)^{\dot\beta}{}_{\dot\alpha}$, where $(M^\dagger)^{\dot\beta}{}_{\dot\alpha} = (M^*)_{\dot\alpha}{}^{\dot\beta}$ reflects the definition of the hermitian adjoint matrix as the complex conjugate transpose of the matrix.

Spinors labeled with one undotted or one dotted index are sometimes called spinors of rank one [or more precisely, spinors of rank $(1,0)$ or $(0,1)$, respectively].

[13] In this book, the dotted-index notation is used in association with the dagger to denote conjugation, as specified in eq. (1.54). In contrast, many references in the supersymmetry literature employ the bar notation made popular by Wess and Bagger [B40], where $\overline{\psi}_{\dot\alpha} \equiv \psi^\dagger_{\dot\alpha} \equiv (\psi_\alpha)^\dagger$.