# 0
# Preliminaries

Algebraic $K$-theory can be understood as a natural outgrowth of the attempt to generalize certain theorems in the linear algebra of vector spaces over a field to the wider context of modules over a ring. We assume the reader has been exposed to the fundamentals of module theory, including submodules, quotient modules, and the basic isomorphism theorems. To establish notation and terminology, we review some definitions in module theory. Then we present an introduction to the language of categories and functors.

By an **additive abelian group** we mean an abelian group $A$ with operation denoted by "+," identity denoted by "$0_A$," and the inverse of an element $x \in A$ by "$-x$." By a **ring** we mean an associative ring with identity — that is, an additive abelian group $R$ with a multiplication $R \times R \to R$, $(r, s) \mapsto rs$, satisfying

$$\begin{aligned} r(s+t) &= rs + rt \ , \\ (r+s)t &= rt + st \ , \\ (rs)t &= r(st) \ , \end{aligned}$$

for all $r, s, t \in R$ and having an element $1 = 1_R \in R$ with

$$1r = r = r1$$

for all $r \in R$.

Suppose $R$ is a ring. A **left** $R$**-module** is an additive abelian group $M$ together with a function $R \times M \to M$, $(r, m) \mapsto rm$, satisfying

$$\begin{aligned} r(m+n) &= rm + rn \\ (r+s)m &= rm + sm \\ (rs)m &= r(sm) \\ 1m &= m \end{aligned}$$

for all $r, s \in R$ and all $m, n \in M$. An $R$**-linear map** $f : M \to N$, between left $R$-modules $M$ and $N$, is a homomorphism of additive groups that also satisfies

$$f(rm) = rf(m)$$

for each $r \in R$ and $m \in M$.

A **right** $R$**-module** is an additive abelian group $M$ together with a function $M \times R \to M$, $(m, r) \mapsto mr$, satisfying

$$\begin{aligned} (m+n)r &= mr + nr \\ m(r+s) &= mr + ms \\ m(sr) &= (ms)r \\ m1 &= m \end{aligned}$$

for all $r, s \in R$ and $m, n \in M$. An $R$**-linear map** $f : M \to N$, between right $R$-modules $M$ and $N$, is a homomorphism of additive groups satisfying

$$f(mr) = f(m)r$$

for each $f \in R$ and $m \in M$.

In either left or right modules we refer to elements of $R$ as **scalars**, elements of $M$ as **vectors**, and the maps $R \times M \to M$ or $M \times R \to M$ as **scalar multiplication**. The set of all $R$-linear maps from $M$ to $N$ is denoted by $\mathrm{Hom}_R(M, N)$. An $R$-linear map $f : M \to M$, from a module $M$ to itself, is called an **endomorphism** of $M$, and $\mathrm{Hom}_R(M, M)$ is denoted by $\mathrm{End}_R(M)$. Our preference will be to work with left $R$-modules — that is, to write scalars on the left side of vectors. So when we refer to an $R$-module with no mention of left or right, we mean a left $R$-module.

Sometimes a vector multiplication is defined from $M \times M$ to $M$. If $R$ is a commutative ring, an $R$**-algebra** $A$ is an $R$-module that is also a ring and satisfies $r(ab) = (ra)b = a(rb)$, whenever $r \in R$ and $a, b \in A$. For example, if $R$ is a subring of a ring $A$, and $ra = ar$ for all $r \in R$ and $a \in A$, then $A$ is an $R$-algebra with scalar multiplication $R \times A \to A$ restricting the ring multiplication $A \times A \to A$.

Whether we study groups, rings, modules, or any other type of mathematical structure, it is often useful to consider the functions which preserve that structure. For instance, the study of groups includes an examination of group homomorphisms. For a comprehensive view of groups, one might consider the class of all groups, together with all group homomorphisms. Imagine a vast diagram of dots connected by arrows, with each group represented by a dot and each group homomorphism by an arrow. What you are picturing is the "category of groups."

**(0.1) Definition.** A **category** $\mathcal{C}$ consists of a class **Obj** $\mathcal{C}$ of **objects**, and for each pair of objects $A, B$ a set **Hom**$(A, B)$ of **arrows** from $A$ to $B$, and for each triple of objects $A, B, C$ a function:

$$\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$$

called the **composite**. The composite of a pair $(f, g)$ is denoted by $g \circ f$. For $\mathcal{C}$ to be a category, the following three axioms must hold:

(i) $\mathrm{Hom}(A,B) \cap \mathrm{Hom}(C,D) = \emptyset$ unless $A = C$ and $B = D$.
(ii) For each object $A$, there is an arrow $i_A \in \mathrm{Hom}(A,A)$ for which

$$j \circ i_A = j \quad \text{and} \quad i_A \circ k = k$$

whenever $j \in \mathrm{Hom}(A,B)$ and $k \in \mathrm{Hom}(C,A)$ for objects $B, C$.
(iii) If $f \in \mathrm{Hom}(A,B), g \in \mathrm{Hom}(B,C)$, and $h \in \mathrm{Hom}(C,D)$, then

$$(h \circ g) \circ f = h \circ (g \circ f) .$$

In a category $\mathcal{C}$, an arrow $f \in \mathrm{Hom}(A,B)$ is said to have **domain** $A$ and **codomain** $B$, and this is implied by the expressions:

$$f : A \to B \quad \text{or} \quad A \xrightarrow{f} B .$$

Axiom (i) just says the domain and codomain of $f$ are uniquely determined by $f$. The arrow $i_A$ in axiom (ii) is called the **identity** arrow on $A$; it is uniquely determined by the composition in $\mathcal{C}$, since if $i$ and $i'$ are two arrows from $A$ to $A$ with the property described in (ii), then $i = i \circ i' = i'$.

The set of arrows $\mathrm{Hom}(A,B)$ is sometimes denoted by $\mathbf{Hom}_{\mathcal{C}}(A,B)$ to emphasize that they are arrows in the category $\mathcal{C}$. We shall often write $\mathbf{End}(A)$ or $\mathbf{End}_{\mathcal{C}}(A)$ to denote the set $\mathrm{Hom}_{\mathcal{C}}(A,A)$ of arrows from $A$ to itself.

**(0.2) Examples.** Here are the names, objects, and arrows of some of the categories considered in this book:

$\mathfrak{Set}$: sets; functions.
$\mathfrak{Group}$: groups; group homomorphisms.
$\mathcal{A}\mathfrak{b}$: abelian groups; group homomorphisms between them.
$\mathfrak{Ring}$: rings (associative with 1); ring homomorphisms (preserving 1).
$\mathfrak{CRing}$: commutative rings (associative with 1); ring homomorphisms (preserving 1) between them.
$\mathfrak{Top}$: topological spaces; continuous maps.
$\mathfrak{Metric}$: metric spaces; isometries.
$\mathfrak{Power}(S)$: subsets of the particular set $S$; inclusion maps ($i : A \to B$, $i(a) = a$) between them.

For each ring $R$ we have the categories:

$R$-$\mathfrak{Mod}$: left $R$-modules; $R$-linear maps between them.
$\mathfrak{Mod}$-$R$: right $R$-modules; $R$-linear maps between them.

For each commutative ring $R$ we have the category:

$R$-$\mathcal{A}\mathfrak{lg}$: $R$-algebras; $R$-linear ring homomorphisms between them.

For the advantage of brevity, when $\mathcal{C}$ is a category, we shall sometimes write $A \in \mathcal{C}$ instead of $A \in \mathrm{Obj}\ \mathcal{C}$ to indicate that $A$ is an object of $\mathcal{C}$. For instance,

$A \in \mathcal{CRing}$ means $A$ is a commutative ring, and $A \in R\text{-}\mathcal{Mod}$ means $A$ is a left $R$-module. This is really an abuse of notation, since the category $\mathcal{C}$ is not the same as the class Obj $\mathcal{C}$; and we will use this shortcut only where it does not suggest any ambiguity.

Each of the preceding examples are **concrete** categories, meaning that the objects are sets (possibly with additional structure), the arrows are (structure-preserving) functions between those sets, the identity arrows are identity functions ($i(x) = x$), and the composition is the composition of functions ($(g \circ f)(x) = g(f(x))$). But many useful categories do not fit this description. Here are some non-concrete categories:

**(0.3) Examples.**

(i) The objects and arrows of a category need not represent anything. For example, we can construct a category with two objects $A$ and $B$, and four arrows $i$, $j$, $f$, and $g$:

$$i \ \circlearrowright A \underset{g}{\overset{f}{\rightleftarrows}} B \circlearrowleft \ j \ .$$

Since the codomain of $f$ equals the domain of $g$, there must be a composite $g \circ f$ from the domain $A$ of $f$ to the codomain $A$ of $g$. Since $i$ is the only arrow from $A$ to $A$, we must have $g \circ f = i$. In this way, the scarcity of arrows forces the composites to be defined by the table:

| $\circ$ | $i$ | $j$ | $f$ | $g$ |
|---|---|---|---|---|
| $i$ | $i$ | | | $g$ |
| $j$ | | $j$ | $f$ | |
| $f$ | $f$ | | | $j$ |
| $g$ | | $g$ | $i$ | |

Each of the sets Hom$(A, A)$, Hom$(A, B)$, Hom$(B, A)$, and Hom$(B, B)$ has only one element. So the category axioms (ii) and (iii) hold automatically, since both sides of each equation belong to the same set Hom$(X, Y)$.

More generally, any diagram with the following properties has exactly one law of composition making it into a category:

(a) For each object $X$ there is an arrow from $X$ to $X$.
(b) If there is an arrow from $X$ to $Y$ and an arrow from $Y$ to $Z$, there is an arrow from $X$ to $Z$.
(c) For each pair of objects $X$ and $Y$ (possibly equal) there is at most one arrow from $X$ to $Y$.

(ii) Suppose $A$ is a set with a partial order $\prec$. That is, $A$ has a binary relation $\prec$ that is reflexive (for all $x \in A$, $x \prec x$), antisymmetric ($x \prec y$ and $y \prec x$ imply $x = y$), and transitive ($x \prec y$ and $y \prec z$ imply $x \prec z$). View the elements of $A$ as objects, and draw one arrow from $x$ to $y$ if $x \prec y$, and no arrow from $x$ to $y$ if $x \not\prec y$. Then properties (a), (b), and (c) of the preceding example hold; so this diagram is a category. We use the term **poset** to refer to a partially ordered set, or to its associated category.

(iii) If $M$ is a set with a binary operation $\circ$ that is associative with an identity element $e \in M$, then $M$ is called a **monoid**. For each object $A$ in any category $\mathcal{C}$, the set $\text{End}_{\mathcal{C}}(A)$ is a monoid under composition of arrows. Every monoid $(M, \circ)$ can be obtained in this way: Just create a category with one object, called $A$, and with the elements of $M$ regarded as the arrows from $A$ to $A$; and define the composition of these arrows by using the operation $\circ$ in $M$.

(iv) Suppose $R$ is a ring. There is a category $\mathcal{M}at(R)$ whose objects are the positive integers 1,2,3,..., and in which $\text{Hom}(m, n)$ is the set of $m \times n$ matrices

$$A = (a_{ij}) = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

with entries $a_{ij}$ in $R$. The composition is matrix multiplication: $B \circ A = AB$. Explicitly, if $A = (a_{ij})$ is an $m \times n$ matrix and $B = (b_{ij})$ is an $n \times p$ matrix, their product $AB = (c_{ij})$ is the $m \times p$ matrix with $ij$-entry

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} .$$

From this formula one can show that the multiplication of matrices is associative, and there is an identity $I_n = (\delta_{ij}) \in \text{Hom}(n, n)$ for each $n$, where:

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j . \end{cases}$$

For details, see (1.26) below.

**(0.4) Definition.** A category $\mathcal{D}$ is called a **subcategory** of a category $\mathcal{C}$ if every object of $\mathcal{D}$ is an object of $\mathcal{C}$, and for each pair $A, B \in \text{Obj } \mathcal{D}$, $\text{Hom}_{\mathcal{D}}(A, B) \subseteq \text{Hom}_{\mathcal{C}}(A, B)$, and composites and identities in $\mathcal{D}$ agree with those in $\mathcal{C}$. In case $\text{Hom}_{\mathcal{D}}(A, B) = \text{Hom}_{\mathcal{C}}(A, B)$, for each pair $A, B \in \text{Obj } \mathcal{D}$, the subcategory $\mathcal{D}$ is called **full**.

Note that if $\mathcal{C}$ is a category, every subclass $\mathcal{D} \subset \text{Obj } \mathcal{C}$ is the class of objects of a full subcategory $\mathcal{D}$ of $\mathcal{C}$. In particular, $\mathcal{A}b$ is a full subcategory of $\mathcal{G}roup$, and $\mathcal{CR}ing$ is a full subcategory of $\mathcal{R}ing$. However, if a set $S$ has at least two elements, the subcategory $\mathcal{P}ower(S)$ of $\mathcal{S}et$ is not full.

Is $\mathcal{R}\mathfrak{ing}$ a subcategory of $\mathcal{A}\mathfrak{b}$? After all, every ring $(R, +, \cdot)$ is an additive abelian group $(R, +)$ if we forget its multiplication, and every ring homomorphism is a homomorphism between additive groups. The answer is *no:* $\mathcal{R}\mathfrak{ing}$ is *not* a subcategory of $\mathcal{A}\mathfrak{b}$! For the condition in the definition of subcategory, "every object of $\mathcal{D}$ is an object of $\mathcal{C}$," is to be taken quite literally: A ring has two operations and an abelian group has only one. So $(R, +, \cdot)$ is not the same as $(R, +)$.

**(0.5) Definition.** In each category $\mathcal{C}$ an arrow $f \in \text{Hom}(A, B)$ is called an **isomorphism** if there is an arrow $g \in \text{Hom}(B, A)$ with

$$f \circ g \;=\; i_B \quad \text{and} \quad g \circ f \;=\; i_A \;.$$

Such an arrow $g$ is called an **inverse** to the arrow $f$. An isomorphism in $\text{End}(A)$ is called an **automorphism** of $A$, and the set of automorphisms of $A$ is denoted by $\mathbf{Aut}(A)$ or $\mathbf{Aut}_{\mathcal{C}}(A)$.

In the categories $\mathcal{S}\mathfrak{et}$, $\mathcal{G}\mathfrak{roup}$, $\mathcal{A}\mathfrak{b}$, $\mathcal{R}\mathfrak{ing}$, $\mathcal{C}\mathcal{R}\mathfrak{ing}$, $R$-$\mathcal{M}\mathfrak{od}$, $\mathcal{M}\mathfrak{od}$-$R$, and $\mathcal{M}\mathfrak{etric}$, an arrow is an isomorphism if and only if it is bijective. In $\mathcal{T}\mathfrak{op}$, an isomorphism is the same as a homeomorphism; but not every bijective continuous map is a homeomorphism. (Consider the identity function on $X$ with two different topologies, the first finer than the second.)

In $\mathcal{M}\mathfrak{at}(R)$, an isomorphism in $\text{Hom}(n, n)$ is the same as an invertible $n \times n$ matrix over $R$. There exist rings $R$ for which $\mathcal{M}\mathfrak{at}(R)$ includes an isomorphism in $\text{Hom}(m, n)$ even though $m \neq n$. Such an isomorphism would be an $m \times n$ matrix $X$ over $R$ for which there is an $n \times m$ matrix $Y$ over $R$ with

$$XY \;=\; i_m \quad \text{and} \quad YX \;=\; i_n \;,$$

where $i_m$ and $i_n$ are identity matrices of different sizes. But such rings $R$ are a little hard to come by. For an example, see (1.37).

**(0.6) Definition.** An object $X$ in a category $\mathcal{C}$ is called **initial** if, for each object $A \in \text{Obj}\ \mathcal{C}$, there is one and only one arrow in $\text{Hom}(X, A)$. An object $Y$ in $\mathcal{C}$ is called **terminal** if, for each object $A \in \text{Obj}\ \mathcal{C}$, there is one and only one arrow in $\text{Hom}(A, Y)$.

**(0.7) Proposition.** *If objects $X$ and $Y$ of a category $\mathcal{C}$ are both initial or both terminal, then there is one and only one arrow $f : X \to Y$, and $f$ is an isomorphism in $\mathcal{C}$.*

*Proof.* The existence and uniqueness of $f$ is immediate since $X$ is initial or $Y$ is terminal. Likewise there is a unique arrow $g : Y \to X$, and the identity arrows are the only arrows in $\text{End}(X)$ and $\text{End}(Y)$. So, for lack of options, $g \circ f = i_X$ and $f \circ g = i_Y$. ∎

**(0.8) Examples.**

(i) In $\mathfrak{Set}$ the empty set $\emptyset$ is initial: $\mathrm{Hom}(\emptyset, A)$ has one member $f : \emptyset \to A$ with empty graph. Each set $\{x\}$ with only one member is terminal.
(ii) In $\mathfrak{Group}$, any trivial group $\{e\}$ is both an initial and a terminal object.
(iii) In $\mathfrak{Ring}$, $\mathbb{Z}$ is an initial object, and the trivial ring $\{0\}$ is a terminal object. The same is true in $\mathfrak{CRing}$.
(iv) For each scalar ring $R$, the trivial $R$-module $\{0\}$ is both initial and terminal in $R$-$\mathfrak{Mod}$.
(v) If $R$ is a ring with at least two elements, $\mathfrak{Mat}(R)$ has no initial object and no terminal object, since each set $\mathrm{Hom}(m, n)$ has more than one element.

Once categories are viewed as algebraic entities, it is natural to ask what might be meant by a "homomorphism" from one category to another. A function is an arrow in $\mathfrak{Set}$; its domain and codomain are sets. As a generalization, define a **metafunction** from a class $A$ to a class $B$ to be a procedure that assigns to each member $a \in A$ a unique member $f(a) \in B$. Then a homomorphism between categories should be a metafunction that takes objects to objects, arrows to arrows, and respects domains, codomains, composites, and identities.

**(0.9) Definition.** A **covariant functor** $F : \mathcal{C} \to \mathcal{D}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a metafunction that assigns to each object $A$ of $\mathcal{C}$ an object $F(A)$ of $\mathcal{D}$, and to each arrow $f : A \to B$ in $\mathcal{C}$ an arrow $F(f) : F(A) \to F(B)$ in $\mathcal{D}$, so that

(i) $F(i_A) = i_{F(A)}$ , and

(ii) $F(g \circ f) = F(g) \circ F(f)$ ,

whenever $A \in \mathrm{Obj}\ \mathcal{C}$ and $(f, g)$ are composable arrows in $\mathcal{C}$.

There is also an arrow-reversing version:

**(0.10) Definition.** A **contravariant functor** $F : \mathcal{C} \to \mathcal{D}$ is a metafunction that assigns to each object $A$ in $\mathcal{C}$ an object $F(A)$ in $\mathcal{D}$, and to each arrow $f : A \to B$ in $\mathcal{C}$ an arrow $F(f) : F(B) \to F(A)$ in $\mathcal{D}$, so that

(i) $F(i_A) = i_{F(A)}$ , and

(ii) $F(g \circ f) = F(f) \circ F(g)$ ,

whenever $A \in \mathrm{Obj}\ \mathcal{C}$ and $(f, g)$ are composable arrows in $\mathcal{C}$.

For both covariant and contravariant functors $F$, the condition (ii) amounts to saying that if $F$ is applied to every object and arrow in a commutative triangle in $\mathcal{C}$, one obtains a commutative triangle in $\mathcal{D}$ (but with its arrows reversed in the contravariant case).

**(0.11) Examples.**

(i) If $\mathcal{C}$ is a subcategory of $\mathcal{D}$, the **inclusion functor** $F : \mathcal{C} \to \mathcal{D}$ is defined by $F(A) = A$ and $F(f) = f$ for each object $A$ and arrow $f$ in $\mathcal{C}$. The inclusion functor $F : \mathcal{C} \to \mathcal{C}$ is called the **identity functor** $i_{\mathcal{C}}$.

(ii) If $\mathcal{C}$ is a concrete category, such as $\mathfrak{Group}$ or $\mathfrak{Ring}$, the **forgetful functor** $F : \mathcal{C} \to \mathfrak{Set}$ is defined by taking $F(A)$ to be the underlying set of elements of $A$ and $F(f)$ to be $f$ as a function. There are also functors involving a partial loss of memory, such as the functor $F : \mathfrak{Ring} \to \mathcal{A}b$ that forgets multiplication: $F(R) = R$ as an additive group and $F(f) = f$ as an additive group homomorphism.

(iii) If $n$ is a positive integer, there is a functor

$$M_n : \mathfrak{Ring} \to \mathfrak{Ring}$$

where $M_n(R)$ is the ring of $n \times n$ matrices with entries in $R$; and if $f : R \to S$ is a ring homomorphism, $M_n(f) : M_n(R) \to M_n(S)$ is the ring homomorphism defined by

$$M_n(f) \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} = \begin{bmatrix} f(a_{11}) & \dots & f(a_{1n}) \\ \vdots & & \vdots \\ f(a_{n1}) & \dots & f(a_{nn}) \end{bmatrix} .$$

(iv) If $R$ is a ring, the elements of $R$ with (two-sided) multiplicative inverses in $R$ are called **units**. The set $R^*$ of units in $R$ is a group under multiplication. There is a functor

$$U : \mathfrak{Ring} \to \mathfrak{Group}$$

defined by $U(R) = R^*$ and $U(f) = f$ restricted to units. This works because every ring homomorphism $f$ takes units to units.

(v) If $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ are functors, there is a composite functor $G \circ F : \mathcal{C} \to \mathcal{E}$ defined by

$$G \circ F(A) = G(F(A)) , \quad G \circ F(f) = G(F(f)) ,$$

for objects $A$ and arrows $f$ in $\mathcal{C}$. If $F$ and $G$ are both covariant or both contravariant, then $G \circ F$ is covariant; if one of $G, F$ is covariant and the other is contravariant, then $G \circ F$ is contravariant.

(vi) The composite of $M_n : \mathfrak{Ring} \to \mathfrak{Ring}$, followed by $U : \mathfrak{Ring} \to \mathfrak{Group}$, is the functor

$$GL_n = U \circ M_n : \mathfrak{Ring} \to \mathfrak{Group} ,$$

which takes each ring $R$ to the group $GL_n(R)$ of $n \times n$ invertible matrices with entries in $R$, and takes each ring homomorphism $f$ to the group homomorphism that applies $f$ to each entry. The $GL$ stands for "general linear" group.

(vii) In the functor definitions (0.9) and (0.10), axioms (i) and (ii) hold automatically if $\mathcal{D}$ is a poset category, for in a poset category an arrow is uniquely determined by its domain and codomain. Also, the class of objects in a poset category is a set. So a functor from a poset $(A, \prec)$ to a poset $(B, \prec)$ is just a function $F : A \to B$ that preserves order:

$$x \prec y \;\Rightarrow\; F(x) \prec F(y)$$

if $F$ is covariant, or reverses order:

$$x \prec y \;\Rightarrow\; F(y) \prec F(x)$$

if $F$ is contravariant.

If $F : \mathcal{C} \to \mathcal{D}$ is a functor and there is a functor $G : \mathcal{D} \to \mathcal{C}$ for which

$$F \circ G = i_{\mathcal{D}} \quad \text{and} \quad G \circ F = i_{\mathcal{C}} \,,$$

then $G$ is called an **inverse** to $F$, and $F$ is called an **isomorphism** of categories if $F$ is covariant and an **anti-isomorphism** of categories if $F$ is contravariant. If there is an isomorphism (respectively, anti-isomorphism) of categories from $\mathcal{C}$ to $\mathcal{D}$, we say $\mathcal{C}$ and $\mathcal{D}$ are **isomorphic** (respectively, **anti-isomorphic**).

The correspondence theorems of algebra provide either isomorphisms or anti-isomorphisms of poset categories: For instance, if $A$ is a group with a normal subgroup $N$, the poset of subgroups of $A$ containing $N$ is isomorphic to the poset of subgroups of $A/N$; the isomorphism takes each $H$ to $H/N$, and its inverse takes each subgroup $K$ of $A/N$ to its union $\cup K$. As a contravariant example, if $F \subseteq E$ is a finite-degree Galois field extension, the poset of intermediate fields is anti-isomorphic to the poset of subgroups of $\mathrm{Aut}(E/F)$; the anti-isomorphism takes each $K$ to $\mathrm{Aut}(E/K)$, and its inverse takes each subgroup $H$ of $\mathrm{Aut}(E/F)$ to the fixed field $E^H$.

For each ring $R$, right $R$-modules are, in a sense, mirror images of left $R$-modules. Their definitions are parallel, and so are the theorems that apply to them. But it is also possible to pass through the looking glass: If $R$ is a ring, define its **opposite ring** $R^{op}$ to have the same elements as $R$ and the same addition as in $R$, but to have a multiplication $\cdot$ defined by $r \cdot s = sr$ (the right side multiplied in $R$).

**(0.12) Proposition.** *There are isomorphisms of categories:*

$$R\text{-}\mathfrak{Mod} \;\cong\; \mathfrak{Mod}\text{-}R^{op} \,,$$
$$R^{op}\text{-}\mathfrak{Mod} \;\cong\; \mathfrak{Mod}\text{-}R \,.$$

*Proof.* If $M$ is a left $R$-module with scalar multiplication $R \times M \to M$, $(r, m) \mapsto r * m$, then the additive group of $M$ can also be made into a right $R^{op}$-module via a scalar multiplication $M \times R^{op} \to M$, $(m, r) \mapsto m\#r$, defined by $m\#r = r * m$:

$$\begin{aligned} (m+n)\#r &= r*(m+n) = (r*m)+(r*n) = (m\#r)+(n\#r) , \\ m\#(r+s) &= (r+s)*m = (r*m)+(s*m) = (m\#r)+(m\#s) , \\ m\#(s\cdot r) &= (rs)*m = r*(s*m) = (m\#s)\#r , \\ m\#1 &= 1*m = m . \end{aligned}$$

Each $R$-linear map $f : M \to N$ between left $R$-modules is also an $R^{op}$-linear map between right $R^{op}$-modules:

$$f(m\#r) = f(r*m) = r*f(m) = f(m)\#r .$$

So there is a functor $F$ from $R$-$\mathfrak{Mod}$ to $\mathfrak{Mod}$-$R^{op}$ with $F((M,*)) = (M,\#)$ and $F(f) = f$.

Similarly, there is a functor $G$ from $\mathfrak{Mod}$-$R^{op}$ to $R$-$\mathfrak{Mod}$ with $G((M,\#)) = (M,*)$, where $r*m$ is defined to be $m\#r$, and $G(f) = f$. Since $F$ and $G$ are inverses, $F$ is an isomorphism. Since $(R^{op})^{op} = R$, the second isomorphism follows from the first. ■

So, for the study of properties held in common by all module categories, it suffices to consider only a category $R$-$\mathfrak{Mod}$ of left $R$-modules. Of course, if $R$ is commutative ($R^{op} = R$), then we can regard $R$-$\mathfrak{Mod}$ and $\mathfrak{Mod}$-$R$ as the same category. Every additive abelian group is a $\mathbb{Z}$-module in exactly one way, and every homomorphism between abelian groups is $\mathbb{Z}$-linear. So $\mathbb{Z}$-$\mathfrak{Mod}$ $= \mathfrak{Mod}$-$\mathbb{Z}$ $= \mathcal{Ab}$, and $\mathbb{Z}$-$\mathcal{Alg} = \mathcal{Ring}$.

Sometimes the values of two functors from $\mathcal{C}$ to $\mathcal{D}$ are closely related in $\mathcal{D}$. For instance, $GL_n$ and $U$ are functors from $\mathcal{CRing}$ to $\mathcal{Group}$, and the determinant connects them: For each commutative ring $R$, the determinant is a group homomorphism $\det : GL_n(R) \to U(R)$. And if $f : R \to S$ is a ring homomorphism, the square

$$\begin{array}{ccc} GL_n(R) & \xrightarrow{\det} & U(R) \\ {\scriptstyle GL_n(f)}\downarrow & & \downarrow{\scriptstyle U(f)} \\ GL_n(S) & \xrightarrow[\det]{} & U(S) \end{array}$$

commutes in $\mathcal{Group}$; so det is compatible with a change of rings.

**(0.13) Definition.** If $F$ and $G$ are covariant functors from $\mathcal{C}$ to $\mathcal{D}$, a **natural transformation** $\tau : F \to G$ is a metafunction that assigns to each object $X$ of $\mathcal{C}$ an arrow

$$\tau_X : F(X) \to G(X)$$